Cantor K.O.'d −− again!

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This is a modified (new! improved!) version of my recent proof demonstrating why "infinite" lists are logically impossible. Cantorists assume, axiomatically, that such lists can be *defined* into existence: but definitions must be free of internal inconsistency, if the defined object is to be something tenable and not a preposterous canard.

Let L be a list indexed to natural numbers, such that: (I) Each list member, n, consists of n symbols; (II) For m from 1 to n, the m_th symbol of the n_th list member is a circle with m dots inside.

Thus, the first list member has one symbol: a circle with one dot inside. The second member has two symbols: a circle with one dot, followed by a circle with two dots. The third member has three symbols: a circle with one dot, then a circle with two dots, then a circle with three dots. Subsequent list members follow the same plan.

This plan ensures that each subsequent list member contains a concatenation of all previous members' terminal symbols.

The list contains an imbedded diagonal string; each of its symbols is the terminal symbol of a list member.

The first symbol of the diagonal is identical to the first list member; the first two symbols of the diagonal are identical to the second list member; the first three symbols of the diagonal are identical to the third list member; and in general the first n symbols of the diagonal are identical to the n_th list member.

By starting with a list consisting of one member, then adding additional members one by one, as the list grows in length, the length of the list members also grows, and the length of the diagonal grows with them; and it can be seen that the diagonal must grow *exactly* as the list members do, otherwise the diagonal cannot grow. The diagonal is a derivative entity, mirroring the growth of the list members.

Is it possible for such a list to cover the range of natural numbers, N? This is equivalent to asking whether it is possible to have an L such that: (a) The quantity of list members is "infinite"; (b) The length of the diagonal is "infinite"; (c) The length of the members remains finite.

Call this hypothetical list L*. If it exists, it may be regarded as a kind of limiting case for the process of list growth just described. In L* the diagonal is still made of the terminals of list members; however, the diagonal must now contain all possible terminal symbols.

It may not always be explicitly recognized that a list is intrinsically a geometric entity: but this is the case, when each member is a string of symbols; each symbol occupies an ordinal location in an abstract space, and possesses spatial relations of "right" and "left" relative to its neighbors; and different members occupy similar ordinal locations, only their spatial relations, relative to one another, are those of "above" and "below"; and all of this implies that individual symbols and list members are separated by spaces so as to preserve their status as distinct entities. Thus, a "list" is a geometric structure. Metrics, however, being arbitrary, are rarely considered.

It is clear that L*, if it exists, can be mapped into a unit square in the plane. One such scheme, using standard orthogonal coordinates, maps each symbol according to the following formulae:

x = 1 − 1/2^(m−1)
y = −1 + 1/2^(n−1)

It is understood that the mapping specifies these points as the centers of the circular symbols.

Thus, the symbols of the diagonal, running from upper−left to lower−right, take the coordinates:
(0,0) ; (1/2,−1/2) ; (3/4,−3/4) ; (7/8,−7/8) ; etc.

The coordinates of the unit square's corners are:
UL (0,0) ; UR (0,1) ; LL (0,−1) ; LR (1,−1). It can be seen that except for the Origin at the upper−left, these are limit points and do not have list symbols mapped to them.

The diameters of the circles can be specified according to any arbitrary rule that permits them to fit without overlap. Since the intervals between mapping points (circular centers) diminish according to the terms of a convergent sequence as the borders of the unit square are approached, thus accomodating an unlimited number of symbols, the circular diameters must also diminish accordingly.

The formula d = 1/2^(n+m) (where d is the diameter) should suffice.

Why specify such details? Because the engineers at Infinite Constructs, Ltd. require them. Not content with the hand−waving of Cantorist set−theorists who presume to "define" in one stroke what may not be possible to exist, I have hired this most reputable firm (licensed and bonded) to actually construct the list L*; in a supervisory capacity we will be taking a mind's−eye tour, travelling along the diagonal line connecting (0,0) with (1,−1), stopping at each of the diagonal's mapping points at each stage of its construction, to make sure the job is done right. Our average travel speed is a leisurely square−root−of−two units per hour: thus we shall traverse the entire diagonal in one hour. Those who maintain that it is impossible to traverse an infinite number of points in a finite time should consult Zeno; and those who remain skeptical should take this opportunity to announce their rejection of point−set geometry and the calculus.

Our little day−hike of infinity may thus be considered illustrative of the adage solvitur ambulando. (Despite Randy Poe's hysterical invective to the contrary, this does not properly translate as "proof by foot−stomping", even if it does entail a kind of forced march.)

Our tour originates at the Origin. Here, the first symbol of the diagonal is laid. It is also, of course, the first (and only) symbol of the first list member. At this point, the diagonal and the first list member are identical.

On we go: The second symbol of the diagonal is laid. The diagonal, now consisting of two symbols, is identical to the two symbols of the second list member. That is, in both length and content, the diagonal and the second list member are identical.

Here we are at the third stage. The diagonal and the third list member are now identical. The pattern is clear. It is, I fear, rather monotonous. We shall skip ahead some considerable way...

Here we are at the 1,013_th stage. The diagonal is identical, in both content and length, to the 1,013_th list member. Goodness, this is boring! When *will* the diagonal become infinite, thus distinguishing itself from the list members, all of which must remain finite? Hmmm...that's a toughie. The Cantorists would probably answer: "When the list is completed". We shall see.

Meanwhile, *how* can the diagonal ever become infinite, since at each stage n the diagonal is identical to the n_th list member, and every possible n is a finite natural number?

The diagonal is a derivative entity, and merely mirrors the list members from which it is derived. It *should* therefore be clear that in order for the diagonal to become infinite and contain all possible symbols, some eventual list member must itself become infinite and contain all possible symbols. After all, the diagonal we are constructing can grow no faster −− and no longer −− than the list members from which it is derived. Well, let's address the matter again when we are done...

..And here we are, after one hour of travel, at coordinates (1,−1), the bottom−right corner of the unit square. Have we not constructed every member of L* using our straightforward mapping formulae?

Did we ever reach a stage of the list at which the list members became infinite? No, because by design each list member was indexed to a natural number, and every natural number is finite. Did the diagonal whose construction we were supervising, stage by stage, ever become infinite, then? No, because at *every* stage the diagonal mirrored the list at that point. No infinite list member, no infinite diagonal.

Because the diagonal intersects and mirrors the list at every stage, the only way the diagonal could be infinite would be for a list member to be infinite, and for the two to meet at the lower−right corner of the unit square (1,−1), where we are now. Here −− and nowhere else −− where our infinite list is at last COMPLETE, the "last" list member (the infinite one identical to the completed infinite diagonal) would have to have its "last" symbol: naturally enough this would have to consist of a circle with an "infinite" number of dots inside it.

Of course, there is no "last" list member; and each member, being by design indexed to a natural number, must contain a finite number of symbols; therefore, a member with an "infinite" number of symbols is not possible; and a symbol consisting of a circle with an "infinite" number of dots inside it is not a possible symbol for our list. And, of course, the diameters of the circular symbols have been decreasing as the limit point at (1,−1) has been approached: at the point (1,−1) any such circle would have to have a diameter of zero, and a "circle of diameter zero" is also a contradiction in terms.

This leads to another point. I have said that only here at the limit point (1,−1) can the list, and the diagonal, be complete: at no point in the half−open interval leading up to it can the diagonal be complete, since any such point is succeeded by additional mapping points further along the diagonal path, and all such mapping points are indexed to finite natural numbers. But the diagonal is not mapped to the limit point (1,−1); therefore nowhere, and at no time, does "the complete diagonal" exist. Furthermore, the lengths of the partial segments of the diagonal, from (0,0) to (1/2,−1/2), then from the latter to (3/4,−3/4), then from the latter to (7/8,−7/8), etc., would form an infinite convergent series whose sum is the square−root of two: therefore the length of the diagonal is also the square−root of two and the diagonal must terminate in the limit point (1,−1); but it cannot, since no symbol of the list (hence no symbol of the diagonal) is mapped to that point.

The solution, of course, is that L* does not exist, and therefore cannot be mapped into a unit square. It does not exist because its very postulation entails logical inconsistency. L* poses as an entity consisting of discrete parts which is simultaneously "limitless" and "completed": but there is no such entity because that is a contradiction in terms.

One cannot have "all" of something that is definitionally limitless. In fact, the term "natural numbers" does not even specify a set of individual numbers, but merely references a particular *concept* of number; a set of rules or template through which individual, specific, actual entities can be created. There is no {N}. There are an unlimited number of {n}, each of which is finite. N is merely a general criterion for classification, much like "individuals named Smith". There is no "set of all Smiths" because there is no theoretical limit to the number of Smiths; one can keep constructing new Smiths as long as the original Smith, or his progeny, continue to procreate and pass the name Smith through their lineage. Similarly, new natural numbers can always be created and named, whose lineage can be traced, directly or indirectly, back to the first natural number, 1. Thus, "Smiths" refers to a general concept of name and to the rules for passing it on, just as "naturals" refers to a general concept of number and to the rules relating its examples through direct or indirect reference to the number 1.

Finally, the author wishes to verify that he does indeed himself reject point−set geometry and the calculus, as well as Santa Claus and The Tooth Fairy. Those who believe that it is possible to have, much less pass through, an "infinite" (never−ending) sequence of points, are delusional, as are those who imagine that the 19th century development of "convergent infinite series" provides a mathematical solution to intractable logical problems. (In fact, that "development" merely re−invokes those problems.)

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