On Well−Ordering(s) and Sets Dense in the Reals, Infinity

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In some recent discussion about the well−ordering of sets dense in the reals, concepts about the specificities of a "previous" and "next" points on the real number line broaden and solidify.

Where the rationals and irrationals, eg numbers of the form a/b for integer a, b as rational numbers and otherwise as irrational numbers are complements the union of which is the set of real numbers and as well each is dense in the real numbers, then a naive rationalization of a given point, as a real or scalar number, and its immediate neighborhood in terms of the finite set of points that are having the least difference, with having rationals and irrationals alternate on the real number line is that: a naive rationalization of the immediate neighborhood of a point on the continuous real number line.

The irrationals may well be divided into two sets, each dense in the irrationals and complements whose union is the irrationals: the algebraic and transcendental irrational numbers, where the algebraic irrationals are real solutions or roots of a polynomial with integer coefficients that are not otherwise rational, and the transcendentals are then the other irrationals.

Then, there are considerations of other ways to divide the sets that are thus far to be determined to be separate, in that their intersection is null, and complementary in that their ultimate union is the set of reals which have the property of completeness and on the line continuity, and dense in the reals where between any two definitely valued numbered reals there are infinitely many members of these sets: rationals, algebraic irrationals, and transcendentals.

To that end it is useful to determine further categorizations of these subsets of the reals that share these three properties of being nonintersecting, complementarily forming the reals, and each dense in each other and the reals.

One possible consideration is that there are many ways to reorganize or compose from the above sets' definitions other definitions of sets that can be said to comprise the reals. Where the above definitions might seem to be the most distinct in the sense of immediacy, there are further subcategorizations of the transcendentals as for example Mahler's S, T, and U transcendentals, if they are disjoint and dense in the reals. As well, something like the rationals can be separated into, for example, the rationals with even versus the rationals with odd denominator, each dense in the reals with their union the rationals. As well, where definitions as above form a tree structure rooted at the reals, various subsets could be composed together in their definition with for example the transcendentals unioned with the rationals or rationals with even denominator unioned with the irrationals: there are many ways to determine two, three, or more of the nonintersecting, dense, completing subsets of the reals to be of many various definitions.

Thus where a naive rationalization of the reals as alternating rationals and irrationals from their properties of being nonintersecting, dense(in each other and the reals), and completely forming in their union the entire set of reals, those properties hold for many other combinations of sets with those properties.

That naive rationalization might be along the lines that along the real number line that rationals q in Q and irrationals p in P alternate as so:

....pqpqpqpqpqpqpqpqp...

with the notion that given a selected distinct rational q, that the immediate neighbor in that contiguous sequence is _not_ an element of Q, and is in this case an element of P, the only other set considered.

Towards that rationalization if there are, for example, rationals q in Q, algebraic irrationals in a in A, and transcendental irrationals t in T, then that naive rationalization follows the the exact specific number that is greater than a given q in Q and less than any other specific number is _not_ an element of Q, but that it be either an algebraic irrational or transcendental number. This is where the real numbers are well−ordered by fiat or theorem. _If_ the next number is an a, then the pattern thus follows:

...qatqatqatqat...

else

... qtaqtaqtaqta

Here, there should be a replacement of those letters that represent specific set with the N, C, D properties for nonintersecting (disjoint), completing, and dense, with a given number of digits that represent how many of those sets are deemed to exist.

...012012012012...
...021021021021...

One problem with that is that the rationals, for example, can be broken into, for example, numbers with even and odd denominators, or numbers with denominators equal to 0, 1, 2 (mod 3), or 0, 1, ..., n−1 (mod n). Thus the rationals can be divided into n many NCD subsets, for n E Z+. The algebraics can as well be broken into various definitions of subsets with these properties, and the transcendentals may be possibly divided into further subsets in these ways.

When there are four or more subsets, then the specific "next" element is not predetermined by previous elements. That is because where there is the negative condition that a successive element _not_ be of the same categorization of the current, nor as possible previous, there are more possibilities under those constraints implied by each's density within each other.

...012301230...

It would seem that that order would be fixed for any definition of these sets numbered 0, 1, 2, and 3 with the NCD property. That is not to say the order could not be:

...01320132...
...02130213...
...02310231...
...03210321...
...03120312...

but once a given permutation of (0, 1, 2, 3) was determined, it would hold through all successions of those values. Equivalently, the sets could be relabelled but the order would always be:

...01230123...

It might seem that for n > 3 that the order could vary, the fugue. By fugue I mean that type zero elements would be each n'th element, but the others would be arbitrary in their permutation except for that the immediate neighbors of the type 0 elements would differ.

...0123013203120321023102130...

If you've read this far, then you might consider that the rationals and irrationals alternate in the reals. Then again, by the same naive rationalization algebraic and transcendentals would alternate, etcetera etcetera for any pair of disjoint sets dense in each other, as ordered fields, whose complement is the reals.

The vague fugue continues, and further methods to subcategorize NCD sets would be a way to further explore this question: in the well−ordered reals, what is the next?

It is regularly claimed that there is no "next" or "previous" due to abscence of well−ordering. Where there is well−ordering, there is.

Besides the moduli of the denominator of the rational, also may be considered various combinations to do with the coprimality of that modulus of the numerator. For example: even and odd denominators and even and odd numerators for the odd numbers. Combinatorially, the number of ways to divide the rationals into these disjoint sets each disjoint, dense in each other, and complementarily forming as their union the rationals, explodes.

The algebraic irrationals have even further combinatoric possibilities as a function of any number of finite integers as the order of the polynomial increases past two, the least order of a polynomial with possibly algebraic as opposed to necessarily rational roots. Indeed, the least order to represent the polynomial the roots of which is a given algebraic irrational is one possible jumping−off point to asymptotically compare that variety.

The transcendentals again, as roots of power series for example with no finite order, or more exploredly as non−algebraic irrationals, see categorizations such as Liouville's and Mahler's S, T, and U types, are readily subdivided.

That is all so, yet once again the real numbers are rationals and irrationals, or algebraics and transcendentals, and where any of these N, C, D sets is everywhere discontinuous, where the goal here is to determine a specification of a point and finitely many of its nearest, least different neighbors, all the possible nonintersecting, completing, dense sets alternate.

That is a function of their density and disjointness, when infinitely many of the contiguous elements of a well−ordered sequence of the reals has that due to the density of each within the reals that deductively finitely many of the contiguous elements would seem to _intuitively_ alternate. To consider finitely many elements of the sequence might indeed lead to a deductive breakdown, where as there are infinitely many ways to divide the reals into NCD sets that any finite sequence would only have some few elements, and perhaps only one of each. Yet, where the reals are comprised (a union) of only some finitely many sets, then a finite sequence could contain one of each, again in each subsequence of n elements one of the n subdivisions.

This discussion is self−contained among my various other arguments, it's specifically about this. Some choose to not even address the concept.

That's irrelevant, here are some questions: in what ways may the rationals or generally algebraics, or transcendentals or generally irrationals, be divided into nonintersecting (disjoint) sets each dense in their union? Is it possible to describe and parameterize all possible ways that they can be so subdivided?

Between any two rationals a/b and c/d, there is an irrational. Between any two irrationals you might define as a decimal (ie Dedekind or Cauchy), there is a rational.

(Infinite sets are equivalent.)

Infinite sets are equivalent for other reasons, eg induction shows that an infinite set is inexhaustible, the binary case is sufficient and certain composable monotonic mappings avoid contradictory functions, and there can only be one proper class. Some theories do have a set of all sets.

This consideration then of how to informedly guess what the "next" element in the well−ordered reals would be in terms of its characteristic properties besides being infinitesimally different and "one−sided" at this point seems to be that into however many sets you subdivide the reals, that many alternatives cycle.

Consider the Nyquist limit, and the hyperintegers. Consider sampling a real by flipping a coin infinitely many times, and how that might be a sample of finitely or infinitely many reals at once.

I think we all learned in high school math that the rationals and irrationals can not be said to alternate on the real number line, perhaps with little justification, that's trusted. It was covered in passing.

So, what do they do?

There are lots of rational numbers.

1. In what ways may the rationals or generally algebraics, or transcendentals or generally irrationals, be divided into nonintersecting (disjoint) sets each dense in their union?

2. Is it possible to describe and parameterize all possible ways that they can be so subdivided?

3. So, what do they do?

These deductions may be valid, the ones expressed here are not particularly controversial, they are couched in inquisitive terms, but then the question arises: what would be the use of such analysis?

You might know that I talk of iota as the "least positive real", ie, the actual number right after zero in the well−ordering of the reals is named iota. As well, iota would be the difference between any given real number and its previous and next in that nonstandard real number model, and its successive elements are named "2iota", "3iota", etcetera. In that sense 0iota = 0 and n*iota converges to 1 as n diverges, per the natural/unit equivalency function.

Speaking of iota itself, where zero is rational, iota is only known to be _not_ rational. Where zero is algebraic, iota is _not_ algebraic. For any of the NCD sets that contain zero, except for trivially changing the definition about zero, iota does not have the property that defines that set.

In "trivially changing" the definition, for example dividing the reals into rationals besides zero and irrationals and zero, then iota, or "the next element after zero in the well−ordering of the reals", would appear to be non−zero and as well rational, where again the negation stems from the density, complementarity, and disjointness of those two specified sets. There could also be deductive curiousities about subdivisions of the rationals. For example if the rationals are divided into reduced fractions with even and odd denominators, then iota is simply irrational, and zero is algebraic. The more "simple" definition would have more meaning.

There is to be no defining the sets by their containment of iota, iota's definition itself rests upon those sets' definition without iota.

Iota exists if you well−order the reals.

Returning to the notion of the rationals and irrationals and their place in the reals, the "simple" delineation or categorization of the reals seems prima facie to be either the separation into rationals and irrationals or the separation into algebraics and transcendentals. In either case the progression is as so:

...01010101010101...

Not much use can be made of a presupposed fact that the rationals and irrationals alternate in the well−ordered reals, because those sets of numbers are fields. It is only that that fact can be presupposed for applications that do not conflict with their characteristics as NCD sets for the reals in the well−ordering of the reals, via fiat in ZFC, where any set can be well−ordered.

So, in well−ordering the reals, the "next" element after zero is the infinitesimal, irrational, and real: iota.

Where that may be so, simply assigning a name to each element of the well−ordering of the unit interval, there still remains a deductive impasse about the density of these sets that comprise the reals, and the notion of the interspersal of each within each other on the finite scale of contiguous elements of the sequence derived from well−ordering the set of reals.

Where due ot their density, in addressing only the rationals and irrationals, each is dense in the reals thus:

i. between any two rationals a/b and c/d there exist infinitely many irrationals
ii. between any two irrationals x E N^N and y E N^N (more precisely x E P and y E P, P < N^N, many sequences converge to the same value) there exists infinitely many rationals

Then a question is whether in the examination of a finite subsequence (of length > 2) of the well−ordering whether there exists in any finite subsequence of the well−ordering an irrational between each pair of rationals and an irrational between each pair of rationals. If there exists only irrationals on some finite subsequence, then not all irrationals can be defined by Dedekind or Cauchy, and those definitions are insufficient to define each real number. If each finite subsequence that contains irrationals contains rationals then the rationals and irrationals are equivalent.

There is no finite well−ordering sequence between any two rationals a/b and c/d, is thus the well−ordering principle contradictory in assumption? Well−ordering is a consequence of Choice without which the reals are not a set.

So, you can divide the reals into infinitely many nonintersecting, dense in the reals sets whose complement is the reals, but the reals are only rationals and irrationals. That's not to say they aren't algbraics and transcendentals, they're rationals and irrationals.

Infinite sets are equivalent (equipotent, equipollent).

Under what conditions can it be said the reals are well−orderable? In ZFC, any set is well−orderable. If any set of reals dense is well−orderable then so is the complete superset and any subset. Is not that obvious to you?

If a set is well−orderable, then its elements can be iterated.

Infinite sets are equivalent.

Regards,

Ross Finlayson

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"The key, of course, is to synthesize a context around each and all of the quotes."