Re: A Manual of Practical Reason

Source: http://sci.tech−archive.net/Archive/sci.logic/2005−01/2092.html

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From: Chris Menzel (*cmenzel_at_remove−this.tamu.edu*)
Date: 01/24/05

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Date: 24 Jan 2005 20:11:12 GMT
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On 24 Jan 2005 10:06:04 −0600, Acme Diagnostics
<LFinezapthis@partpostmark.net> said:
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> poopdeville@gmail.com wrote:
>>Acme Diagnostics wrote:
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>>This is silly, Larry. If you're really interested in knowing whether
>>or not popular presentations of Godel's theorem accurately communicate
>>the result, you should pick up a good textbook, read through the
>>theorem, and figure out why the version presented is misleading.
>
> We already covered that in the argument on this
> subject which ended on 6−20−04 with this post:
> http://tinyurl.com/44yvh (Your side lost. <g>)

I don't plan to engage in any further debate here, Larry, as I think
this particular horse has been thoroughly whipped to death, but
re−reading the masterpiece again reminded me just how many problems it
suffers from. The following if from that URL (thanks for making it tiny
;−) with a one−sentence preface from you:

> > [Larry:]
> > My compilation of Kent Paul Dolan's posts, with two unauthorized
> > revisions to substitute "axiom system" for "set of axioms:"
> > [Dolan revised:]
> [Goedel] proved that any set of axioms at least as rich as the
> axioms of arithmetic

There is no such thing as "the" axioms of arithmetic. It is critical to
expressing the theorem correctly that one speak about particular systems
of arithmetic. What Dolan's trying to talk about here are axiom systems
that are capable of proving (i.e. that are "at least as rich as") a
certain minimal amount of arithmetic (which, in more technical
treatments, can be characterized very precisely).

> has statements which are true in that axiom system, but cannot be
> proved by using that set of axioms.

As has been stated again and again, the idea of a sentence being "true in a set of axioms" is nonsense. A sentence can be true in a model, and a sentence can be provable from a set of axioms, but the author has apparently conflated these two ideas. He either goofed, or simply doesn't have a clear understanding of the basics of mathematical logic.

> *That does not prevent that those true things can be proved*
> *with a more powerful set of axioms. It only conveys that the*
> *stronger axiom system will in turn contain new truths which*
> *cannot be proved using only those axioms.*

This is a bit closer to the mark, but it still shows the same confusion between truth and proof. The first sentence is ok, but the second goes on to say that the stronger system will "contain new truths" that it cannot prove. First off, there is no common notion in logic of a system of axioms "containing a truth". Granted, one can imagine a couple of ways of defining such a notion −− notably, we could define a system to contain a truth if it can prove it! −− but no notion I can think of makes sense of the sentence in question. What Dolan is missing here is the distinction between sentences that are *expressible* in the vocabulary of a system (notably, a vocabulary containing symbols for addition and multiplication) and the sentences *provable* in the system. And what he's trying to get at here is that we cannot escape incompleteness by moving to stronger systems that can prove things that are not provable in weaker systems. The stronger systems will have "gaps" of their own.

> *Goedel's incompleteness theorem only shows that some true*
> *math facts cannot be proved within math, not that none of them*
> *can.*

Again, Dolan's wandering around in the right neighborhood but doesn't quite have the address (though who would have thought that no "truth math facts" can be proved within math?). The context of Godel's theorem is arithmetic, and while it generalizable to any system that contains a bit of arithmetic, it is best to keep that context explicit instead of talking generally about "math facts". Second, (and rather trivially), any arithmetical truth *is* in fact "provable in math", in at least one clear sense. Suppose system S doesn't prove statement A. It is a simple theorem of logic that both A and not−A can be added (separately, of course) to S and the result is consistent; that is, both S + A and S + not−A are consistent. But either A is true or not−A is true. So either way, we will have a consistent system that proves the truth in question (whichever one it is) "within math" −− though we may never *know* which of the systems S + A or S + not−A (if either) is the true system. What we can be sure of by Godel's theorem is that we will never have a *single* axiomatic system that encompasses all arithmetical (hence, of course, mathematical) truths. Again, it is the failure to relativize to particular systems of arithmetic, and to speak instead of what is "provable in math" per se, that lies at the root of the unclarity here.

> *It isn't all that complicated to follow the proof, either, since it*
> *uses only the axioms of arithmetic to achieve its goal.*

This is egregiously false. In fact, the proof requires some fairly high−powered recursion theory, set theory, and model theory.

As always, I wish you the best, but if these facts don't speak for themselves to you, then we just have nothing whatever to discuss.

Chris Menzel

ps: If you want an informal version of Godel's theorem, you could do worse than to replace Dolan's version with the following (please note I will not argue its merits over Dolan's; this is just a friendly suggestion):

For any consistent axiomatic system capable of proving a certain minimal amount of arithmetic, there will be sentences expressible in the vocabulary of the system that can be neither proved nor disproved in the system.

Note the theorem doesn't actually say anything about truth at all −− indeed, the prominence of truth in Dolan's version is a further, more global problem with it −− though of course we can immediately infer:

Corollary: For any consistent axiomatic system capable of proving a certain minimal amount of arithmetic, there will be truths of arithmetic expressible in the system that the system cannot prove.