Re: .999... = 1

Source: http://sci.tech−archive.net/Archive/sci.logic/2005−08/msg00496.html

- *From*: David C. Ullrich <ullrich@xxxxxxxxxxxxxxxx>
- *Date*: Mon, 29 Aug 2005 06:11:42 −0500

On 28 Aug 2005 21:37:41 −0700, "MoeBlee" <jazzmobe@xxxxxxxxxxx> wrote:

>.999... = 1. As famous as this theorem is, I haven't been able to find
>a rigorous proof (one that does not use infinite strings in additive
>columns as if the strings were finite) and I haven't been able to prove
>it for myself. I'd like to have a proof that:
>
>lim of SUM[k = 1 to n]9/(10^k) = 1.

Assuming "lim" means the limit as n −> infinity then yes,
that's exactly what "0.999... = 1" means, so it's exactly
what you want to prove.

>I tried the following (since this is such a familiar subject, such
>things as that 'e' ranges over reals and 'n' over naturals are
>implicit):
>
>Let f(n) = SUM[k = 1 to n]9/(10^k)
>
>Show that, for all e, there exists n, such that, for all k > n, |1 − e|
>> |1 − f(k)|.

But this is not the definition of limit. You need to prove this:

(*) For any e > 0 there exists n such that for all k > n

|1 − f(k)| < e.

Note there are two differences there, regarding what e is and
what inequality is required.

Now how do you prove (*)? There are two ingredients:

(i) f(k) = 10^(−k).

(ii) For any e > 0 there exists a positive integer n such that

10^(−n) < e.

>From what you've written here it seems clear that you should have no trouble seeing how (i) and (ii) imply (*); the problem with your attempt at a proof was you had the definition of "limit" wrong.

And you also should have no trouble giving a proof of (i) by induction. What about (ii)?

In many expositions (ii) is just skipped over as obvious. But (ii) does require proof − for example there exist "ordered fields" (_roughly_, structures where _algebra_ works exactly the same as it does for the real numbers) where (ii) is _false_!

What's needed to prove (ii) is the "completeness axiom". It takes a minute to say what that is:

Say S is a set of real numbers and b is a real number. We say b is a lower bound for S if b <= x for every x in S. We say S is bounded below if there exists a lower bound for S. The completeness axiom is this:

Axiom: If S is a (nonempty) set of reals and S is bounded below then S has a greatest lower bound.

Here "greatest lower bound" means exactly what it says: a lower bound which is greater than every other lower bound.

For the rest of this post let

S = {10^(−n) : n is a posiitive integer}.

Then S is bounded below, for example 0 is a lower bound for S. So S has a greatest lower bound. In fact:

(iii) The greatest lower bound of S is 0.

If we know (iii) then (ii) follows: If e > 0 and 0 is the greatest lower bound for S then e is not a lower bound for S, and (ii) says exactly "if e > 0 then e is not a lower bound for S".

To prove (iii): We know that 0 is a lower bound for S. Suppose that 0 is not the greatest lower bound for S. Then there exists b > 0 such that b is a lower bound for S. But it's easy to see that if b is a lower bound for S then 10b is also a lower bound for S, and if b > 0 then 10b > b.

So: If S has a strictly positive lower bound
then S cannot have a greatest lower bound
(contradiction), because given any positive
lower bound b we can find a larger one, namely 10b.
So S does not have a strictly positive lower
bound, which says exactly that 0 is the
greatest lower bound for S.

QED.

>Suppose |1 − e| > 1/(10^j) for some j > 0. Then let n = j. Let k > n.
>Show, by induction on k, that 1/(10^j) > |1 − f(k)|.
>
>For the basis step, k = 0, the result is vacuously true, since k > n >
>0. Or, the basis step could have k = 2 (since k > n > 0) and, in this
>case, the basis step holds since if k = 2, then j = 1, and 99/100 is
>closer to 1 than is 9/10. But at the inductive step I just get tangled
>in calculations that don't seem to be leading to the result, as well as
>I do not see how to use the crucial fact that 9 is the numerator. Maybe
>I'm approaching this wrong from the start.
>
>This is not an exercise, as I just would like to have this proof as a
>matter of record for a conversation about the subject. So, while hints
>are appreciated, a finished proof would be even more appreciated. I
>hope the proof doesn't require much more than the little I know, but if
>the proof requires some more advanced theorems, then so be it, as I'll
>need to learn these theorems anyway.
>
>Thanks in advance for any help you would provide.
>
>MoeBlee
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>P.S. This is posted to sci.logic rather than sci.math, since I just
>happen to be familiar with the postings of enough people in sci.logic
>that I know there are some who I can trust their expertise.

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- ***Follow−Ups***:
  - ***Re: .999... = 1***
    - *From:* MoeBlee
  - ***Re: .999... = 1***
    - *From:* Ken Quirici

- ***References***:
    - ***.999... = 1***
        - *From:* MoeBlee

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