Re: What is the 1st order formal system known as PA?

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- *From*: "Rupert" <rupertmccallum@xxxxxxxxx>
- *Date*: 20 Nov 2005 18:49:38 −0800

MoeBlee wrote:
> Rupert wrote:
> > This is an informal presentation of Peano's axioms, probably modelled
> > on the presentation Peano himself originally gave. It's different to a
> > formal presentation.
>
> It is much more than a difference in formality. The theory shown at the
> web site, which is in the same vein as Peano's own formulation is a
> much weaker and less expressive theory (for lack of '+' and '*', which
> I don't think I'm mistaken in thinking that they are not definable from
> just '0' and 'S' along the few axioms at that website) than what we've
> agreed is PA. And that web site is hardly an exception.
>

In that case, I would expect that the formal counterpart of the theory
they are expounding is second−order Peano arithmetic, in which addition
and multiplication can be defined by recursive definitions.

> > > So Chang & Keisler don't include the seventh and eighth axioms of
> > > Shoenfield. And I don't think those axioms are derivable, even with the
> > > definition of '<', from the first six axioms along with axiom nine, are
> > > they?
> >
> > Yes, they are.
>
> Yep, you're right, and trivial too. Thanks. So Shoenfeld gives
> superfluous axioms. Hmm.
>

They're not superfluous if you're taking "<" as a primitive term. They
are superfluous if you take it as a defined term.

> So here's what we agree is first order PA:
>
> First order logic with identity and language:
>
> 0

> S
> +
> *
>
> Axioms:
>
> (1) Sn not= 0
>
> (2) Sn=Sk −> n=k
>
> (3) n+0=n
>
> (4) n+Sk=S(n+k)
>
> (5) n*0=0
>
> (6) n*Sk=(n*k)+n
>
> (7) (phi[0] & An(phi(n) −> phi[Sn])) −> An phi[n]
>

Yep.

> > > Right, I don't know all the details of that, but I have a sense of the
> > > outline. However, what I don't understand is that within set theory we
> > > show that any two complete ordered fields are isomorphic with one
> > > another. Yet, Lowenheim−Skolem tells us that there are non−isomorphic
> > > models of the axioms,
> >
> > Which axioms?
>
> (1) (x+y)+z = x+(y+z)
>
> (2) x+y=y+x
>
> (3) Ey x+y=z
>
> (4) (x*y)*z = z(x*y)
>
> (5) x*y=y*x
>
> (6) x+y not= y −> Ez x*z=y
>
> (7) x*(y+z) = (x*y)+(z*z)
>
> (8) (x<y & y<z) −> x<z
>
> (9) x not< x
>
> (10) x<y v y<x v x=y
>

> (11) x<y −> x+z<y+z
>
> (12) Theorem: E!xAy y+x=y
>

I'd be interested to see the proof of this theorem.

> (13) Definition: 0 = the x such that Ay y+x=y
>
> (14) (x<y & 0<z) −> x*z<y*z
>
> (15) Schema: (Ew phi[w] & ExAy(phi[y] −> (y<x v y=x))) −>
> Ez(phi[z] & Ay(Ax(phi[y] −> (y<x v y=x)) −> y not< z))
>

I have my doubts about whether you really can prove the uniqueness of
additive and multiplicative inverses from those axioms. But if you can,
or if we take those facts as axioms, your axioms generate precisely the
theory of real−closed fields. It will *not* be true that every model is
a complete ordered field.

>
> If we were in set theory, so that all the axioms were relativized to a
> certain sets S, and alternatively to a set S' and phi[x] were taken as
> xe{y| yeS & phi[y]} and as xe{y| yeS' & phi[y]}, respectively, then we
> prove that S with its operations (+ on S, * on S) and ordering (< on S)
> is isomorphic to S' with its operations (+ on S', * on S') and ordering
> (< on S').
>

Not true. This would only be true if you replaced your first−order
axiom schema by a second−order version of the least upper bound
principle. As things stand, your axioms have nonisomorphic models.

> Yet, in the meta−theory, which is itself set theory (a copy, up a
> level, of the set theory in the object language?), there are
> non−isomorphic models of the axioms, even though, within set theory in
> the object language, any sets satisfying the conditions embodied in the
> axioms are isomorphic.
>

Hmmm. I thought when you said "if we were in set theory" you were
proposing to take set theory as the metatheory. Now it looks like you
were proposing to take set theory as the object theory. That's
incoherent. The object theory is the first−order theory you were
discussing. That's different to set theory. Only the metatheory can be
set theory. And when we take set theory as the metatheory, as I said,
we prove that your theory has nonisomorphic models. It's only the
second−order version of the theory that has a unique model up to
isomorphism.

> So, maybe I've just reiterated the "paradoxical" aspect of
> Lowenheim−Skolem? Have I just reiterated, in a long around way, that a
> set is evaluated differently from within set theory than it is in the
> metatheory?
>

I can't see how what you're saying is making any sense. There's an object theory, a first−order theory which you presented. For the metatheory we can take set theory. And then we prove that the object theory has non−isomorphic models. That's all there is to it. If for our object theory we had the usual second−order theory, we could prove that the object theory has a unique model up to isomorphism, but that's a different object theory. There's no contradiction.

> > You seem to be assuming that there is a set of first−order axioms such
> > that any model of them is a complete ordered field. This is not the
> > case.
>
> So what I need to say instead is that in set theory, any set that
> satisfies the axioms above is a complete ordered field if we modify
> them to be set−theoretical without the schema (as I described) but that
> not all models of the axioms in their original form, with the schema,
> are complete ordered fields, right?
>

If you take the second−order version of the axioms, then every model is a complete ordered field, but regarding the first−order theory you were discussing, not every model is a complete ordered field.

> Thanks,
>
> MoeBlee

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- ***Follow−Ups***:
  - ***Re: What is the 1st order formal system known as PA?***
    - *From:* MoeBlee
- ***References***:
  - ***What is the 1st order formal system known as PA?***
    - *From:* Nam Nguyen
  - ***Re: What is the 1st order formal system known as PA?***
    - *From:* David C . Ullrich
  - ***Re: What is the 1st order formal system known as PA?***
    - *From:* MoeBlee
  - ***Re: What is the 1st order formal system known as PA?***
    - *From:* Rupert
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- ◊ *From:* MoeBlee
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