# Re: What is the 1st order formal system known as PA?

***Source:*** http://sci.tech−archive.net/Archive/sci.logic/2005−11/msg00445.html

* *From*: "Rupert" <rupertmccallum@xxxxxxxxx>
* *Date*: 21 Nov 2005 01:18:07 −0800

MoeBlee wrote:
> Rupert wrote:
> > They're not superfluous if you're taking "<" as a primitive term. They
> > are superfluous if you take it as a defined term.
>
> But it's definable. I like the short list much better.
>
> > I have my doubts about whether you really can prove the uniqueness of
> > additive and multiplicative inverses from those axioms.
>
> I'll try to remember to double check the arguments. They're in
> Browder's 'Mathematical Analyis'.
>
> > > If we were in set theory, so that all the axioms were relativized to a
> > > certain sets S, and alternatively to a set S' and phi[x] were taken as
> > > xe{y| yeS & phi[y]} and as xe{y| yeS' & phi[y]}, respectively, then we
> > > prove that S with its operations (+ on S, * on S) and ordering (< on S)
> > > is isomorphic to S' with its operations (+ on S', * on S') and ordering
> > > (< on S').
>
> > Not true. This would only be true if you replaced your first−order
> > axiom schema by a second−order version of the least upper bound
> > principle. As things stand, your axioms have nonisomorphic models.
>
> Don't forget, I'm not talking about models for a language. I'm talking
> about structures within set theory.
>
> Here's the statement in Browder:
>
> Theorem. There exists a complete ordered field R. If R1 and R2 are
> complete ordered fields, there exists an isomorphism of ordered fields
> between them, i.e., there exists a bijective mapping psi:R1−>R2 which
> preserves the structure: for any x,yeR1 we have psi(x+y)=psi(x)+psi(y)
> and psi(x*y)=psi(x)*psi(y); for any x,yeR1 with x<y, we have
> psi(x)<psi(y).
>
> He's not in formal set theory, but he might as well be since there's

> nothing he's doing that can't be cast in set theory.
>

Yes, but your hypothesis was just that S and S' are models of your
first−order theory. (Or if not, you should have stated it more
clearly). His hypothesis is that R1 and R2 are complete ordered fields,
which is stronger. His statement is correct; yours is not.

> > > Yet, in the meta−theory, which is itself set theory (a copy, up a
> > > level, of the set theory in the object language?), there are
> > > non−isomorphic models of the axioms, even though, within set theory in
> > > the object language, any sets satisfying the conditions embodied in the
> > > axioms are isomorphic.
> > >
>
> > Hmmm. I thought when you said "if we were in set theory" you were
> > proposing to take set theory as the metatheory. Now it looks like you
> > were proposing to take set theory as the object theory. That's
> > incoherent. The object theory is the first−order theory you were
> > discussing. That's different to set theory. Only the metatheory can be
> > set theory. And when we take set theory as the metatheory, as I said,
> > we prove that your theory has nonisomorphic models. It's only the
> > second−order version of the theory that has a unique model up to
> > isomorphism.
>
> What I mean is that set theory is a first order object level theory,
> call it Z0. And the first order formal meta theory is set theory, call
> it Z1. Z1 is just like Z0 but a level up. Z0 is first order set theory
> at the object level and Z1 is first order set theory at the meta−level.
> Why is that not allowed?

It's allowed, but what do you want to do that for? I would have thought
your object theory would be the first−order theory you were talking
about and your metatheory would be something like set theory.

> What would be the formal meta theory for
> object level set theory? Why can't it be formal meta level set theory?

It can be, but why do you want a metatheory for set theory? Why do you
need that?

> It might as well be, since the informal meta theory is set theory.
> Might as well formalize it. (This is not circular; but it is infinite
> escalation, since there is Z2, Z3,....) But when I said "we're in set
> theory", I meant the object level.
>

No, your object theory is the theory you are discussing, which is the
first−order theory you were talking about. Your metatheory is set
theory. You can have a metametatheory if you want, but I don't see the
point.

> > > So, maybe I've just reiterated the "paradoxical" aspect of
> > > Lowenheim−Skolem? Have I just reiterated, in a long around way, that a
> > > set is evaluated differently from within set theory than it is in the
> > > metatheory?
>
> > I can't see how what you're saying is making any sense. There's an
> > object theory, a first−order theory which you presented. For the
> > metatheory we can take set theory. And then we prove that the object
> > theory has non−isomorphic models. That's all there is to it. If for our
> > object theory we had the usual second−order theory, we could prove that
> > the object theory has a unique model up to isomorphism, but that's a
> > different object theory. There's no contradiction.
>
> Don't forget, I'm not using any second order language or second order
> theory. I have first order object language theories, such as Z0
> (Zermelo set theory), PA, axiom sets about fields, etc. Now, I see that
> the so−called field axioms or so−called axioms for groups, etc. can
> also be not axioms, but rather, definitions, in Z0.

Yes, but you can't define a complete ordered field to be a structure
satisfying some first−order theory. There's no first−order theory whose
models are precisely the complete ordered fields.

> Then in Z0 we show
> that any structure (set theory structure, like an algebraic structure;
> not a structure for a language) that satisfies the definition is
> isomorphic to any other structure that satisfies the definition.

What definition are you talking about here?

For the tenth time: it is *not* true that any two models of your
first−order theory are isomorphic. It *is* true that any two models of
the standard, second−order, theory of complete ordered fields are
isomorphic, but Loewenheim−Skolem doesn't apply there.

> That
> prove is in, for example, Browder.
>

Browder's proof is *not* a proof about models of your first−order
theory. It is a proof about complete ordered fields, which are the same
thing as models of a certain second−order theory. The models of your
first−order theory are real−closed fields, not all of them are complete
ordered fields. It is not true that any two models of your first−order
theory are isomorphic. Browder's proof says nothing about this issue,
Browder's proof is talking about complete ordered fields.

> But, reverting back from definitions, I also have a separate first
> order object level theory, not set theory, but still a first order
> theory, that uses the phi[x] style schema. My set theory definition of

> a certain kind of structure is "translatable" back to an object
> language theory in a different language (as primitives it has + * <,
> rather than set theory which has only e, for 'is an element' while + *
> < are defined) in which the definitions are not definitions but rather
> axioms. Let's call this B. B is a first order object level theory with
> the so−called field axioms and the least upper bound principle as an
> axiom schema.
>

This is not a translation of the definition of a complete ordered field. The first−order axiom schema fails to capture the full power of the second−order principle. The class of structures satisfying this first−order theory is a larger class of structures than the class of complete ordered fields. In fact it is the class of real−closed fields.

> Also, I have a formal first order meta theory that is also a first
> order set theory, Z1 (exactly like Z0, except "declared" to be up a
> level. I.e., a formal first order meta theory, that is set theory, for
> a formal first order object level theory that is set theory. In this
> meta theory is where I assign structures for languages and have models.
> So, in this meta theory, I study object theories, including Z0 and B
> and PA. But I see that not all models of B are isomorphic, even though
> all structures in object level Z0 that satisfy the definitions (just
> like the axioms of B, but instead defintions in Z0) are isomorphic.
>

Any two complete ordered fields are isomorphic.

But the class of complete ordered fields is not the same as the class of structures which are models for B. The latter class is larger.

> The situation is that the object level set theory structures are
> isomorphic but the meta level models are not isomorphic. I'm guessing
> that this is a reflection of Lowenheim−Skolem.
>

It's nothing to do between the distinction between the object level and the meta level. Forget about having a metatheory for set theory. Just work in set theory. Set theory is your metatheory, the first−order theory you were talking about is your object theory.

In set theory, we can prove that any two complete ordered fields are isomorphic. But we can also prove that it's not the case that any two models of B are isomorphic. Both these theorems are theorems of your metatheory. There's no contradiction because not every model of B is a complete ordered field. Simple.

> (One more thing: I take the domains of the models and the relations and
> functions of the model to be sets in the meta theory set theory.)
>

Fine.

> > If you take the second−order version of the axioms, then every model is
> > a complete ordered field, but regarding the first−order theory you were
> > discussing, not every model is a complete ordered field.
>
> I understand that. And what I'm saying is that that is parallel to the
> situation in which all the structures are isomorphic in first order
> object level set theory; but not all models are isomorphic in first
> order meta level set theory.

Your object theory is B. Your metatheory is set theory. In the metatheory, all complete ordered fields are isomorphic but not all models of B are isomorphic.

You want to introduce a metametatheory somewhere into the picture. This is irrelevant to what we're discussing. The phenomenon that all complete ordered fields are isomorphic but not all models of B are isomorphic happens already in the metatheory.

> Because one is isomorphism of structures
> in a theory, and the other is non−isomorphism among models that are
> structures for first order languages.

Whether or not an isomorphism exists between two structures is a question that gets decided in the metatheory. The phenomenon we're observing is just due to the fact that not every model of B is a complete ordered field. It's nothing to do with the distinction between two levels of theory.

> Just like, in first order object
> level set theory, the set of reals is uncountable, but in first order
> meta level model theory (expressible within first order meta level set
> theory) there are countable models if there are infinite models.
>

You can have a model of set theory such that a set is uncountable in the model but countable "in the real world". And you can have a model of set theory such that a structure is a complete ordered field in the model but not a complete ordered field "in the real world".

But these considerations are irrelevant to what we are discussing. There are models of B which are not complete ordered fields in any (transitive) model of set theory. What we are talking about is nothing to do with exotic models of set theory which look different from the inside than from the outside. We are just talking about the simple fact that not every model of B is a complete ordered field.

> MoeBlee

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- ***Follow−Ups***:
    - ***Re: What is the 1st order formal system known as PA?***
        - *From:* MoeBlee
- ***References***:
    - ***What is the 1st order formal system known as PA?***
        - *From:* Nam Nguyen
    - ***Re: What is the 1st order formal system known as PA?***
        - *From:* David C . Ullrich
    - ***Re: What is the 1st order formal system known as PA?***
        - *From:* MoeBlee
    - ***Re: What is the 1st order formal system known as PA?***
        - *From:* Rupert
    - ***Re: What is the 1st order formal system known as PA?***
        - *From:* MoeBlee
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        - *From:* MoeBlee
    - ***Re: What is the 1st order formal system known as PA?***
        - *From:* Rupert
    - ***Re: What is the 1st order formal system known as PA?***
        - *From:* MoeBlee
- Prev by Date: ***Re: Finest partition − exercise in Suppes's book***
- Next by Date: ***Re: What is the 1st order formal system known as PA?***
- Previous by thread: ***Re: What is the 1st order formal system known as PA?***
- Next by thread: ***Re: What is the 1st order formal system known as PA?***
- Index(es):
    - ***Date***
    - ***Thread***