# Re: What is the 1st order formal system known as PA?

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- *From*: "Rupert" <rupertmccallum@xxxxxxxxx>
- *Date*: 21 Nov 2005 15:32:34 −0800


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MoeBlee wrote:
> Rupert wrote:
> > Yes, but your hypothesis was just that S and S' are models of your
> > first−order theory. (Or if not, you should have stated it more
> > clearly). His hypothesis is that R1 and R2 are complete ordered fields,
> > which is stronger. His statement is correct; yours is not.
>
> Thank you for your generous replies. I think perhaps you've
> misconstrued me on certain key points, so that perhaps you think I'm
> disagreeing with you, though I am not. But perhaps you are correct that
> I did not state clearly enough. My premise is not that S and S' are
> models. My premise is that they are sets in set theory. My S1 and S2
> are exactly Browder's R1 and R2. There's no conflict among you, me, or
> Browder in this regard.
>
> > > What I mean is that set theory is a first order object level theory,
> > > call it Z0. And the first order formal meta theory is set theory, call
> > > it Z1. Z1 is just like Z0 but a level up. Z0 is first order set theory
> > > at the object level and Z1 is first order set theory at the meta−level.
> > > Why is that not allowed?
>
> > It's allowed, but what do you want to do that for? I would have thought
> > your object theory would be the first−order theory you were talking
> > about and your metatheory would be something like set theory.
>
> Yes, my metatheory for any object thoery is set theory. And if set
> theory itself is the object theory, then my metatheory for it is set
> theory (just one level up).
>
> > It can be, but why do you want a metatheory for set theory? Why do you
> > need that?
>
> Because my object level set theory is in a formal first order language.
> To form that language and to talk about the theory, I need a
> metatheory.
>

Yes, but on this occasion you don't need to talk about that theory.

> > No, your object theory is the theory you are discussing, which is the
> > first−order theory you were talking about. Your metatheory is set
> > theory. You can have a metametatheory if you want, but I don't see the
> > point.
>
> I have a metalanguage and a metatheory to be able to state and prove
> sentences about the object theories.
>
> > > Don't forget, I'm not using any second order language or second order
> > > theory. I have first order object language theories, such as Z0
> > > (Zermelo set theory), PA, axiom sets about fields, etc. Now, I see that
> > > the so−called field axioms or so−called axioms for groups, etc. can
> > > also be not axioms, but rather, definitions, in Z0.
>
> > Yes, but you can't define a complete ordered field to be a structure
> > satisfying some first−order theory. There's no first−order theory whose
> > models are precisely the complete ordered fields.
>
> Since you explained that a few posts ago, I've understood that. I think
> we're in accord here. I'm saying that in set theory, you can define
> 'complete ordered field' and any two complete ordered fields are
> isomorphic with one another. On the other hand, if you "transform"
> those definitions from set theory into axioms for a different theory
> (with a different language) from set theory, call it B, then the class
> of models of B is not the class of complete ordered fields.
>
> > > Then in Z0 we show
> > > that any structure (set theory structure, like an algebraic structure;
> > > not a structure for a language) that satisfies the definition is
> > > isomorphic to any other structure that satisfies the definition.
>
> > What definition are you talking about here?
>
> The definition, in set theory, of 'compelete ordered field'.
>
> > For the tenth time: it is *not* true that any two models of your
> > first−order theory are isomorphic. It *is* true that any two models of
> > the standard, second−order, theory of complete ordered fields are
> > isomorphic, but Loewenheim−Skolem doesn't apply there.
>
> I have not said that they are. We are on exactly the same page. And
> what you said about second order fits too. In set theory, any two
> complete ordered fields are isomorphic. And any two models of second
> order theory of compelte ordered fields are isomorphic. But it is not
> the case that any two models of the first order theory B (axioms for an
> ordered field plus the axiom schema for least upper bound) are
> isomorphic and there are even models of B that are not complete ordered
> fields.
>

> > Browder's proof is *not* a proof about models of your first−order
> > theory.
>
> Exactly. I didn't claim that it is. I was indeed offering it in
> CONTRAST with with a proof about models.
>
> > It is a proof about complete ordered fields, which are the same
> > thing as models of a certain second−order theory. The models of your
> > first−order theory are real−closed fields, not all of them are complete
> > ordered fields. It is not true that any two models of your first−order
> > theory are isomorphic. Browder's proof says nothing about this issue,
> > Browder's proof is talking about complete ordered fields.
>
> Yep. That's what I was getting at.
>
> > > But, reverting back from definitions, I also have a separate first
> > > order object level theory, not set theory, but still a first order
> > > theory, that uses the phi[x] style schema. My set theory definition of
> > > a certain kind of structure is "translatable" back to an object
> > > language theory in a different language (as primitives it has + * <,
> > > rather than set theory which has only e, for 'is an element' while + *
> > > < are defined) in which the definitions are not definitions but rather
> > > axioms. Let's call this B. B is a first order object level theory with
> > > the so−called field axioms and the least upper bound principle as an
> > > axiom schema.
>
> > This is not a translation of the definition of a complete ordered
> > field. The first−order axiom schema fails to capture the full power of
> > the second−order principle. The class of structures satisfying this
> > first−order theory is a larger class of structures than the class of
> > complete ordered fields. In fact it is the class of real−closed fields.
>
> That's why I put "translatable" in quotes. And, as I mentioned, I
> understand the point you are making, and it has given me perspective
> and explanation as I needed it.
>
> > Any two complete ordered fields are isomorphic.
> > But the class of complete ordered fields is not the same as the class
> > of structures which are models for B. The latter class is larger.
>
> Yep, that's what I've learnt.
>
> > > The situation is that the object level set theory structures are
> > > isomorphic but the meta level models are not isomorphic. I'm guessing
> > > that this is a reflection of Lowenheim−Skolem.
>
> > It's nothing to do between the distinction between the object level and
> > the meta level. Forget about having a metatheory for set theory. Just
> > work in set theory. Set theory is your metatheory, the first−order
> > theory you were talking about is your object theory.
>

> Here I disagree. First order theories such as PA, B, the theory of
> groups, etc. have set theory as the metatheory. But first order set
> theory itself also gets a metatheory, which is also first order set
> theory (but a level up).

Yes, but I don't see why we need to refer to a metatheory for set
theory on this occasion.

> But I think my speculation about
> Lowenheim−Skolem is incorrect. I should not have written that after you
> had already explained about real closed fields and complete ordered
> fields. I should have seen from that that it's not just a cardinality
> issue.
>
> > In set theory, we can prove that any two complete ordered fields are
> > isomorphic. But we can also prove that it's not the case that any two
> > models of B are isomorphic. Both these theorems are theorems of your
> > metatheory. There's no contradiction because not every model of B is a
> > complete ordered field. Simple.
>
> I agree except one point, which reflects my feeling that set theory
> itself can be an object theory. We can prove the isomorphism of
> complete ordered fields in both object level set theory and in meta
> level set theory. That redundancy is just a consequence of the fact
> that I view set theory as both an object level theory and as a
> metatheory. This doesn't complicate things as much as you might expect,
> since the object level theory and the metatheory are just like one
> another except one is a level up.
>
> > Your object theory is B. Your metatheory is set theory. In the
> > metatheory, all complete ordered fields are isomorphic but not all
> > models of B are isomorphic.
>
> Right. Additionally, in my way of looking at, in the object level set
> theory, all complete ordered fields are isomorphic.
>
> > > Because one is isomorphism of structures
> > > in a theory, and the other is non−isomorphism among models that are
> > > structures for first order languages.
>
> > Whether or not an isomorphism exists between two structures is a
> > question that gets decided in the metatheory.
>
> I tried to make clear which of the two senses of 'structure' I meant.
> One is a tuple, such as an algebraic structure; the other is a function
> from the parameters of a language into a domain and its relations and
> functions. In object level or meta level set theory (let's just say,
> set theory, since the difference doesn't matter here) we prove
> isomorphisms between structures (in the first sense of the word); in
> model theory (which is part of the metatheory, which is set theory a
> level up) we look at structures (in the second sense of the word) as

> some of these structures are models for a set of axioms or not.
>
> > The phenomenon we're
> > observing is just due to the fact that not every model of B is a
> > complete ordered field. It's nothing to do with the distinction between
> > two levels of theory.
>
> I see that I was probably barking up the wrong tree on that point. I
> see what you're saying.
>
> > > Just like, in first order object
> > > level set theory, the set of reals is uncountable, but in first order
> > > meta level model theory (expressible within first order meta level set
> > > theory) there are countable models if there are infinite models.
>
> > You can have a model of set theory such that a set is uncountable in
> > the model but countable "in the real world". And you can have a model
> > of set theory such that a structure is a complete ordered field in the
> > model but not a complete ordered field "in the real world".
>
> I have a problem with ""real world"".

Well, you shouldn't. Saying something holds "in the real world" is just
saying that it holds without any restriction on the quantifiers.

> The way I think of it is that in
> set theory, as a formal theory, we prove a sentence that we READ as
> saying there are uncountable sets; literally we have a proven formula,
> which in our intuitive, informal interepretation, says there are
> uncountable sets. In our metatheory, we see that Lowenheim−Skolem tells
> us that there is a countable and an uncountable model. So there are
> FORMAL interpretations (by means of structures for the language) some
> of which have countable domains and some of which have uncountable
> domains. (One wrinkle is that to prove the existence of these models,
> our meta level set theory has to be a stronger version of our object
> level set theory, right?). But at the formal meta level, we still read
> the formulas with our intuitive, informal interpretaion, which is
> occuring in the informal meta−meta level. So we formalize the meta−meta
> theory, ad infinitum. So instead of relying on a notion of a real world
> (an abstract mathematical real world), I rely on an infinite escalation
> of meta theory. The advantage of positing a real world (an abstract
> mathematical real world) is simplicity; the disadvantage is that it is
> not formal, or for, non−platonists, simply unacceptable.

You can be a formalist instead of a platonist if you want to be. I
don't mind. But you still have to pick which theory you're working in.
When I say "there are models in which there are uncountable sets which
are countable 'in the real world'", I am saying something which can be
given as a perfectly well−defined theorem of set theory. If you're
working in that theory, you shouldn't have any problem with what I say.

> The advantage
> of an infinite escalation of meta theory is that it is always formal
> and mathematical, or can be, each time we reach up to formalize. The
> disadvantage of an infinite escalation of meta theory is that it keeps
> deferring − there may not be an ultimate settlement of certain
> questions that would be settled by "looking directly" at the "real
> mathematical world" just to see what is or is not the case in that
> world. As I understand, Godel talks about the possiblity that these
> questions may be settled by finding axioms (perhaps large cardinal
> axioms) that settle them. But I don't see why he has any hope that the
> axioms would be evidently true in the real mathematical world when the
> sentences that we want to settle for truth or falsehood are themselves
> not evidently true or false. What hope do we have that a large cardinal
> axiom would be evidently true when the continuum hypothesis, which is
> to be settled by a large cardinal axiom, is neither evidently true or
> false?
>
> > But these considerations are irrelevant to what we are discussing.
> > There are models of B which are not complete ordered fields in any
> > (transitive) model of set theory. What we are talking about is nothing
> > to do with exotic models of set theory which look different from the
> > inside than from the outside. We are just talking about the simple fact
> > that not every model of B is a complete ordered field.
>
> Yes, I think I was on the wrong track there.
>
> Thanks again for your posts. I am trying my best to be clear, within
> reasonable time for composing and editing.
>
> MoeBlee

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- ***Follow−Ups***:
  - ***Re: What is the 1st order formal system known as PA?***
    - *From:* MoeBlee
- ***References***:
  - ***What is the 1st order formal system known as PA?***
    - *From:* Nam Nguyen
  - ***Re: What is the 1st order formal system known as PA?***
    - *From:* David C . Ullrich
  - ***Re: What is the 1st order formal system known as PA?***
    - *From:* MoeBlee
  - ***Re: What is the 1st order formal system known as PA?***
    - *From:* Rupert
  - ***Re: What is the 1st order formal system known as PA?***
    - *From:* MoeBlee
  - ***Re: What is the 1st order formal system known as PA?***

- ◊ *From:* Rupert
- ♦ ***Re: What is the 1st order formal system known as PA?***
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  - ◊ *From:* Rupert
- ♦ ***Re: What is the 1st order formal system known as PA?***
  - ◊ *From:* MoeBlee


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