Re: Subsets of cardinals in a well−ordering

Source: http://sci.tech−archive.net/Archive/sci.logic/2007−05/msg00610.html

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- *Date*: Fri, 25 May 2007 15:50:03 −0700

In article <1179945153.082475.161790@xxxxxxxxxxxxxxxxxxxxxxxxxxxx>, justinpa84@xxxxxxxxx says...

> (from Kunen − exercise 19) Let kappa be an infinite cardinal, let W well−order kappa. Prove there's a subset X of kappa with cardinality kappa on which W agrees with the usual ordering of ordinals.
>
> One way to rephrase this is that any kappa length sequence of distinct ordinals less than kappa has an increasing subsequence of length kappa.
>
> I can prove it for regular cardinals easily enough, but I'm pretty much stumped on proving it for the more general case (although I've reduced it to several statements trying some other strategies, but I think these are deadends).

I'll suggest an argument for the special case kappa = aleph_omega.
I think it's adaptable to the general case, though I haven't looked
at the details.

Assume the result is true for regular cardinals. Let
kappa = aleph_omega and suppose f:kappa −> kappa is a permutation.
We want to show that there is a subset B of kappa such that
|B| = kappa and f|B is order−preserving.

Let X_0 = aleph_0, X_{n+1} = aleph_{n+1} \ aleph_n,
and A(n,k) = {eta : eta in X_n and f(eta) in X_k }

Note that |A(n,k)| <= aleph_k and X_n = Union_k A(n,k).

Since aleph_n = |X_n| is regular, we must have at least one k such
that A(n,k) has power aleph_n. Let g(n) be that k, that is, g is
a function omega −> omega such that |A(n,g(n))| = aleph_n.

g is unbounded, so there is an infinite subset J of omega on which g is order−preserving. Note that if m<n, both m,n in J, then for all eta in A(m,g(m)), xi in A(n,g(n)), we have eta < xi and f(eta) < f(xi). Now apply the result for regular cardinals to get a subset B_n of A(n,g(n)) such that |B_n| = aleph_n, and f|B_n is order−preserving. Finally, piece together the B_n's to get B.

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