

# Re: Enderton problem

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- *From:* David C. Ullrich <ullrich@xxxxxxxxxxxxxxxxxxxx>
  - *Date:* Wed, 05 Dec 2007 18:33:40 -0600
- 

On Wed, 5 Dec 2007 14:19:29 -0800 (PST), Gc <Gcut667@xxxxxxxxxxxx> wrote:

On 5 joulu, 23:18, David Ullrich <ullr...@xxxxxxxxxxxxxxxxxxxx> wrote:

Gc wrote:

On 5 joulu, 18:20, David Ullrich  
<ullr...@xxxxxxxxxxxxxxxxxxxx> wrote:

Gc wrote:

On 5 joulu, 14:08, David C.  
Ullrich  
<ullr...@xxxxxxxxxxxxxxxxxxxx>  
wrote:

On Tue, 4  
Dec 2007  
13:03:16  
-0800  
(PST), Gc  
<Gcut...@xxxxxxxxxxxx>  
wrote:

On  
4  
joulu,  
02:34,  
dpo...@xxxxxxxxxxxxxxxx  
wrote:

If  
you  
have  
Enderton's  
A  
Mathematical

## Re: Enderton problem

Introduction  
to  
Logic  
handy,  
this  
problem  
comes  
from  
number  
7  
in  
chapter  
2,  
section  
6.  
Consider  
a  
language  
with  
a  
two-place  
function  
predicate  
symbol  
 $<$ ,  
and  
let  
 $N$   
 $=$   
 $(N;$   
 $<)$   
be  
the  
structure  
consisting  
of  
the  
natural  
numbers  
with  
their  
usual  
ordering.  
Show  
that  
there  
is  
some  
A  
elementarily  
equivalent

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to  
N  
such  
that  
 $\langle A$   
has  
a  
descending  
chain.  
Okay,  
here's  
what  
I'm  
thinking.  
We  
let  
the  
domain  
of  
A  
be  
the  
set  
 $\{1/$   
n  
:  
n  
in  
the  
natural  
numbers}  
and  
define  
(m,  
n)  
is  
in  
 $\langle A$   
iff  
(n,  
m)  
is  
in  
 $\langle N$ .  
So  
A  
appears  
to  
have  
a  
descending

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chain.  
Now  
I  
need  
to  
show  
that  
A  
and  
N  
are  
elementarily  
equivalent.  
I  
can  
do  
this  
by  
showing  
A  
is  
a  
model  
for  
ThN.  
But...how  
do  
I  
do  
this?  
Enderton  
suggests  
applying  
the  
compactness  
theorem...but  
I'm  
not  
sure  
how  
this  
leads  
to  
showing  
they  
are  
elementarily  
equivalent.

I  
don't

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see  
why  
you  
can't  
just  
take  
the  
non-positive  
integers  
M  
with  
their  
usual  
ordering.  
That  
structure  
is  
isomorphic  
to  
the  
( $\mathbb{N}$ ;  
<),

It is?

Hint: No,  
it's not.

Hint: It's  
also not  
elementarily  
equivalent  
to  $\mathbb{N}$ ,  
because for  
example  
it does not  
have a least  
element,  
while  $\mathbb{N}$   
does.

Apparently you don't know  
anything what you are  
talking about. You  
have language  $(\mathbb{N}, <)$  with  
usual truths about  $\mathbb{N}$   
expressible in this  
language  $(\mathbb{N}, <)$  eg.  $\exists x \forall y$   
 $\text{not}(y < x)$  is valid.  
Morphism means that we  
have a map  $\alpha$  to which  
holds  $\alpha(\exists x \forall y \text{not}(y < x)) = \exists x \forall y \text{not}(y < x)$

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$(\forall y)(\exists x) y < x$ . So in the positive – integers – structure sign  $<$  is interpreted otherwise.

Yes, until the last sentence. In the model  $(\mathbb{N}, <)$ , where  $<$  is the usual ordering"  $<$  is not interpreted "otherwise".

OK. That was a typo. I meant the non-positives.

Hint: A structure that is isomorphic to  $\mathbb{N}$  cannot possibly be the answer to the question, since if it's isomorphic to  $\mathbb{N}$  it does not have a descending chain.

Look, we can of course interpret  $<$  otherwise.  $(\mathbb{N}, <)$  is not a special, canonical interpretation of  $<$ .

The problem was about the theory of  $(\mathbb{N}, <)$ , where  $<$  is the usual ordering.

In our meta theory in which we do model

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theory, we have in  $(M, <)$  a descending chain, where  $M$  are the non-positive integers.

Yes if  $<$  is the usual ordering on the non-positive integers (in which case  $(M, <)$  is not a solution because it's not elementarily equivalent to  $\mathbb{N}$ ).

Yes it is. Exactly the same formulas are valid in it. If  $nRm = (n < m)$  in  $(\mathbb{N}, <)$  and  $aPb = (-n > -m)$  in  $(M, <)$  then there is map  $\alpha$  such that  $nRm = (n < m) = nRm \iff -nP-m = \alpha(n)P\alpha(m) = (-n > -m)$

I have a hard time following that because of the "a" and "b" – they must have something to do with n and m... But never mind.

No if, as you \_said\_,  $<$  on  $M$  is the inverse of the usual ordering.

But it's just how you interpret things!

Look.

First, let me say this: When I write "N" that's an abbreviation for  $(\mathbb{N}, <)$ , where  $<$  is the usual order on  $\mathbb{N}$ .

Now,  $N$  is \_a\_ structure. It has \_an\_ ordering on it, which is \_the\_ interpretation of " $<$ " in  $N$ . The question is about \_the structure  $N$ \_, not about  $\mathbb{N}$  where  $<$  is interpreted differently.

Each model is an interpretation of the language and the axioms of question.

Uh, yes.

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Let's say  $M_1 = (S, <)$ , where  $S$  is the set of non-positive integers and  $<$  is the usual order on  $S$ . Then  $M_1$  is not elementarily equivalent to  $N$ , because  $N$  has a smallest element and  $M_1$  does not.

It depends on the interpretation of " $<$ ". The interpretations in each model of the symbols in the language are each totally independent of the some other model.

Yes. That's exactly why your comment "It depends on how you interpret things!" makes no sense here. In each of those two models we have fixed an interpretation of  $<$ . Other interpretations are irrelevant.

The sentence  $\exists x \forall y \sim (y < x)$  is true in  $N$  and false in  $M$ , so they're not elementarily equivalent. QED.

If they are isomorphic, they are of course elementarily equivalent. You can check this in for example in Hinman's book of logic.

Of course isomorphic models are elementarily equivalent. There have been many different things above called " $M$ ". Which one are we talking about?

The non-positive integers with the usual ordering are not elementarily equivalent to  $N$ . Hence they do not give a solution to the problem.

The non-positive integers, with  $<$  interpreted as the reverse of the usual ordering, of course are isomorphic to  $N$ . Hence they obvious do not give a solution to the problem.

Which one are you talking about right now?

That sentence is either true in  $M$  or false in  $M$  – talking about the existence of  $\alpha$  such that this and how things are interpreted doesn't change the fact that that sentence is true in  $N$  and false in  $M_1$ .

That would lead that  $(M, <)$  wouldn't be a model at all to  $TH(N, <)$ .

Uh, that's correct. It's not.

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You

get from complete theories which have only infinite models that all their models are categorical.

Say  $M_2 = (S, <)$ , where  $S$  is the set of non-positive integers and  $<$  is the reverse of the usual ordering. Then  $M_2$  is elementarily equivalent to  $\mathbb{N}$ . In fact  $M_2$  is isomorphic to  $\mathbb{N}$ , and hence it's clear that  $M_2$  is not a solution to the problem either, since  $M_2$  does not contain a descending chain.

Yes it does. Do you understand this? If  $nRm = (n < m)$  in  $(\mathbb{N}, <)$  and  $aPb = (-n >^* -m)$  in  $(M, <)$  then there is map  $\alpha$  such that  $nRm = (n < m) = nRm \Leftrightarrow \alpha(n)P\alpha(m) = (-n >^* -m)$  this is map  $\alpha(n) = -m$  just is an isomorphism of the two structures.

Much of the notation here makes no sense, for example defining  $\alpha$  by  $\alpha(n) = -m$ . No doubt the problems are just typos, but I can't figure out what you're trying to say.

$M_2$  does not contain a descending chain. That would be a sequence  $n_1, n_2, n_3, \dots$  such that, with the interpretation of  $<$  used in  $M_2$ ,  $n_1 > n_2 > n_3 \dots$ . And that would be a sequence of non-positive integers  $n_1, n_2, \dots$ , such that with the usual interpretation of  $<$ ,  $n_1 < n_2 < \dots$ . And there is no such sequence.

Now you are claiming that isomorphic structures aren't elementarily equivalent.

What?  $M_2$  does not contain a descending chain. It's isomorphic to  $\mathbb{N}$ .  $\mathbb{N}$  does not contain a descending chain either. How do you conclude that I'm claiming what you say?

The theory of  $(\mathbb{N}, <)$  of course doesn't know the difference if the structures are isomorphic or even elementarily equivalent, but the meta theory does. Look, it is like this: meta theory of informal group theory does know the difference between the symmetric group of order 5 and the special linear group  $(2, 5)$  even if the axioms of the groups doesn't. But they are different structures, because the nature of their elements are different and in some different language when we are looking some

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other aspects of them they might not be isomorphic.

True. So what? Does any of that imply (i)  $\exists x \forall y \sim(y < x)$  is false in  $\mathbb{N}$ ,  
(ii)  $\exists x \forall y \sim(y < x)$  is true in  $M_1$ , or (iii)  $M_2$  has a descending chain?

Yes. In set theory that is our meta theory we can prove that  $(M, <)$  has a descending chain. Give me first the axioms of our theory, then I can verify each and every one of them in the model  $(M, <)$ .

This is nonsense. Tell me what the descending chain is. That has nothing to do with axioms in any theory, it just has to do with the definition of "descending chain".

(If you think so, which one of (i), (ii) or (iii) do you think is true? And why? If you think that  $\exists x \forall y \sim(y < x)$  is false in  $\mathbb{N}$ , please show that there exists a natural number smaller than 0. Without any  $\alpha$ 's or "interpretations" – the truth or falsity of (i) depends precisely on whether there exists a natural number less than 0, period. If you think that  $\exists x \forall y \sim(y < x)$  is true in  $M_1$ , please provide a proof that there is a smallest non-negative integer (in the usual ordering). If you think that  $M_2$  has a descending chain, please give an example of a infinite descending chain in  $M_2$ ; note that that would be a sequence of non-negative integers which is increasing in the standard order.)

The interpretation what you are asking is impossible in the  $(M, <)$ . Surely we agree that you can't interpret our theory eg the all truths of  $\mathbb{N}$  expressible in the language of  $(\mathbb{N}, <)$  that way in the model  $(M, <)$ , because that would be contradictory?

Gorble freck.

\*\*\*\*\*

David C. Ullrich

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