

# Re: Conservativity and CH

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- *From:* [kleptomaniac666 @xxxxxxxxxxxx](#)
  - *Date:* Sat, 8 Dec 2007 08:03:29 -0800 (PST)
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On Dec 8, 5:52 am, "Robert E. Beaudoin" <[rbeaud...@xxxxxxxxxxxx](#)> wrote:

kleptomaniac6...@xxxxxxxxxxxx wrote:

On Dec 6, 6:30 am, "Robert E. Beaudoin" <[rbeaud...@xxxxxxxxxxxx](#)> wrote:

kleptomaniac6...@xxxxxxxxxxxx wrote:

I actually have one more question I was wondering about: Is it the case that if for some axiom  $T$ ,  $ZF+T$  is conservative over  $ZF$  for arithmetical statements, then  $ZF+\text{not}T$  is also conservative over  $ZF$  for arithmetical statements?

Let's take it for granted that  $ZF$  is consistent. (Else  $ZF$  entails every arithmetical statement, so any extension of  $ZF$  would be conservative for arithmetical statements.) Let  $A$  be the axiom "there is no measurable cardinal". Then  $ZF + A$  is satisfied by the constructible universe, and so is conservative over  $ZF$  for arithmetical statements. But  $ZF + \sim A$  entails  $\text{Con}(ZF)$ , which is arithmetical and not derivable from  $ZF$  alone.

Robert E. Beaudoin

Yeah I forgot about the relationship between metamathematical statements and arithmetical statements!

## Re: Conservativity and CH

But how about the following wild conjecture: For any non-arithmetical statement  $S$  which is independent of some system  $R$  where the arithmetical part of  $R$  extends minimal arithmetic or  $Q$ , for any system  $T$  which extends  $R$  and proves  $S$ , there is a system  $T'$  which extends  $R$  and disproves  $S$  and has the same arithmetical part as  $T$ .

All the systems at hand are consistent.

If you are willing to accept consistency of ZF plus existence of a measurable cardinal then the example I already gave refutes your conjecture: Again let  $A$  be the axiom "there is no measurable cardinal" (a non-arithmetical statement). Assuming consistency of measurable cardinals this axiom is independent of ZF. Let  $T$  be the theory with axioms  $ZF + A$ ; since the arithmetical consequences of  $T$  are the same as those of ZF they do not include  $\text{Con}(ZF)$ . But as  $ZF \vdash \sim A \rightarrow \text{Con}(ZF)$  any theory extending ZF and having  $\sim A$  as a consequence also has  $\text{Con}(ZF)$  as a consequence.

Robert E. Beaudoin

The conjecture was, there would be a theory  $T'$  extending ZF that disproves  $A$  and has the same arithmetical consequences as  $T$ .

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