Set theory and identity theory

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There's are some questions in my mind now that I don't know how to settle. They're rather fundamental, so I feel some urgency to get them straightened out.

First, let's axiomatize identity theory (for a language with at least one predicate symbol) as folllows (leaving off leading universal quantifiers as I'll do in this post):

Your favorite axioms (and/or rules) of pure first order predicate logic (in the abovementioned language), plus

Axiom: x=x

Axiom schema: If P is an atomic formula and P' is the same as P except y occurs in zero or more places in P' where x occurs in P, then
x=y −> (P −> P')
is an axiom.

We'll call those 'the axioms of identity theory' (though identity theory may be axiomatized in other ways too).

Now consider a set theory such as Z.

We can take Z to be a non−conservative extension of identity theory for the language with just the primitive 'e' (aside from the primitive '='). So this axiomatization has two primitives − '=' and 'e' and the axioms of identity theory, plus the axiom of extensionality, plus the other axioms of Z. Let's call this axiomatization Z1. And ordinarily along with this we would take the fixed semantics that always intepprets '=' as standing for the identity relation on the universe of the model.

Or we can dispense with identity theory and the axiom of extensionality and have instead the pure logical axioms and/or rules and the rest of the axioms of Z plus the following axiom:

Axiom: Az(zex <−> yex) −> Az(xez −> yez).

So this axiomatization has only one primitive − 'e'. We'll call this axiomatization Z2.

Then we conservatively extend the theory axiomatized by Z2 by adding this definitional axiom (which is actually the ordinary axiom of extensionality):

Def: x=y <−> Az(zex <−> yex)

Now this axiomatization has two non−logical symbols 'e' and '='. We'll call this axiomatization Z3.

Now, if we take 'theory' in the sense of 'a set of sentences closed under entailment', then the consequences of Z1 = the consequences of Z3. They are the exact same set of sentences. That is, Z1 and Z3 axiomatize the same theory. In particular both have the theorems:

x=y <−> Az(zex <−> zey)

and

x=y <−> Az(xez <−> yez).

Now, I want to be quite literal and take the consequences of Z3 as a theory in a language with both '=' and 'e' so that a structure for the language gives an interpretation of both '=' and 'e'. Of course, since '=' is defined from a previous axiomatization, it would suffice to interpret just 'e'; but for the purpose of drawing out my concern here, it makes things more visible to also explicitly interpret '='.

But what about the interpretation of '=' when we've used axiomatization Z3? In this case, since we are not "coming from" identity theory, I would think we wouldn't give '=' the special dispensation of a special semantics. In this case, '=' is a defined symbol, and it gets interpreted however it may be interpreted. But then is it the case that for any model of the consequences of Z3, it turns out ANYWAY that '=' gets interpreted as identity on the universe of the model? (I was unable to prove it when I sat down to the problem last night.) What about when the model is standard (i.e., interprets 'e' as the membership relation on the universe)? What about when the model is non−standard (interpretes 'e' as other than the membership relation on the universe)?

Additionally, with such theories as the consequences of Z1 and the consequences of Z3, we get the indiscernibliity of identicals (which is expressed by the axiom schema of identity theory, which is a theorem schema of Z3). But, as is famous, it is not possible in first order to express the identity of indiscernibles (not even in a meta−language for a fist order theory). However with Z1 and Z3 don't we "override" that in the sense that we have:

Az(zex <−> zey) −> x=y.

That tells us that to conclude x=y, we don't even have to exhaust ALL properties expressible by formulas, but rather that just one single property (expressed by the left side fo the above biconditional). For that matter, we get the same result from another theorem we have:

Az(xez <−> yez) −> x=y.

So, to recap, my questions are:

(1) Is it the case that for any model of the consequences of Z3, it turns out ANYWAY that '=' gets interpreted as identity on the universe of the model? What about when the model is standard (i.e., interprets 'e' as the membership relation on the universe)? What about when the model is non−standard (interpretes 'e' as other than the membership relation on the universe)?

(2) Though not precise, is my general take on the identity of indiscernibles in the right vein regarding this particular situation?

Thanks for any help.

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