Re: Set theory and identity theory

Source: http://sci.tech−archive.net/Archive/sci.logic/2008−03/msg00880.html

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- *Date*: Fri, 14 Mar 2008 00:56:11 +0100

On Thu, 13 Mar 2008 16:04:12 −0700 (PDT), MoeBlee <jazzmobe@xxxxxxxxxxx> wrote:

> Or we can dispense with identity theory and the axiom of extensionality and have instead the pure logical axioms and/or rules and the rest of the axioms of Z plus the following axiom:
>
> Axiom: Az(zex <−> yex) −> Az(xez −> yez).
>
> So this axiomatization has only one primitive − 'e'. We'll call this axiomatization Z2.
>
> Then we conservatively extend the theory axiomatized by Z2 by adding this definitional axiom (which is actually the ordinary axiom of extensionality):
>
> Def: x=y <−> Az(zex <−> zey)
>
> Now this axiomatization has two non−logical symbols 'e' and '='. We'll call this axiomatization Z3.

Yes. I discussed that approach with george some time ago.

> [...]
>
> But what about the interpretation of '=' when we've used axiomatization Z3? In this case, since we are not "coming from" identity theory, I would think we wouldn't give '=' the special dispensation of a special semantics. In this case, '=' is a defined symbol, and it gets interpreted however it may be interpreted. But then is it the case that for any model of the consequences of Z3, it turns out ANYWAY that '=' gets interpreted as identity on the universe of the model?

I don't think so. Note that here "x = y" just "means" that Az(zex <−> zey). Now consider a universe (for our model) which contains "decorated" sets (or "colored" sets if you like. Just assume that we have objects with "elements" which in addition have a color). Then we might have two _different_ decorated sets a, b which just have the same elements. In this case we would have

Az(zea <−> zeb).

And hence

a = b

would be satisfied. BUT a and b would NOT be identical (in the usual sense of the word).

> (I was unable to prove it when I sat down to the problem last night.)

I guess the REASON why you COULD NOT succeed is clear now? No?

> Additionally, with such theories as the consequences of Z1 and the consequences of Z3, we get the indiscernibility of identicals (which is expressed by the axiom schema of identity theory, which is a theorem schema of Z3). But, as is famous, it is not possible in first order to express the identity of indiscernibles (not even in a meta−language for a fist order theory).

This is demonstrated for Z3 (at least) above. While we might claim that the "=" of FOPL= actually refers to /identity/; after all we treat it this way in our semantics of FOPL=. (No?)

> So, to recap, my questions are:
>
> (1) Is it the case that for any model of the consequences of Z3, it turns out ANYWAY that '=' gets interpreted as identity on the universe of the model?

Imho the answer has to be /no/. (See argument from above.)

> What about when the model is standard (i.e., interprets
> 'e' as the membership relation on the universe)?

This does not exclude "colored"/"decorated" sets, does it? Of course we are not dealing with "usual" sets any more.

F.

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