Re: Number Theory

Source: http://sci.tech−archive.net/Archive/sci.math.num−analysis/2006−08/msg00108.html

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- *Date*: 3 Aug 2006 02:02:30 GMT

In article <32616738.1154529058857.JavaMail.jakarta@xxxxxxxxxxxxxxxxxxxxxxx>,
Danilo <danilorj@xxxxxxxxx> wrote:

> Find all pairs m, n so that $2^m + 3^n$ is a perfect square.

Wrong newsgroup. Followups set to sci.math.

I'll let someone else find an easy proof that the complete solution set is
$\{ 2^0 + 3^1 , 2^3 + 3^0, 2^4 + 3^2 \}$
Here is an approach that can be used generally for exponential
Diophantine equations of this particular type (meaning: this method
works for the equations for which it works!) We actually find the
solutions to a broader class of problems.

For some even $N > 4$, treat separately the N^2 values of m and n
mod N ; then you're looking for integer solutions to equations like e.g.
$y^2 = 4 x^N + 243 z^N$
with the added stipulation that x be a power of 2 and z be a power
of 3. This is equivalent to finding the rational points $(X, Y) =$
$(x/z, y/z^{(N/2)})$ on the curve $Y^2 = 4 X^N + 3$. That's a nonsingular
curve and is of genus greater than 1 if $N > 4$, and so by Faltings' theorem
it has only a finite number of rational points. It follows that there are
only finitely many integer solutions (m,n) to the original problem.
So in some sense, we could claim we're done.

Unfortunately Faltings' proof is not effective and so there is (today)
no obvious way to determine when a set of solutions is complete.
On the other hand, "small" equations like these are not observed in
practice to have "very large" rational points, so the points we find
with a modestly thorough search are "probably" the only ones there are.
Let's try to carry this out convincingly.

You can rule out some cases with N=8: we can't have n and m both
odd because there's not even a 2−adic solution to $Y^2 = 2^m X^8 + 3^n$
in such cases. Several other combinations have no rational solutions

except at X=0 or the points at infinity (1/X = 0), because the elliptic
curve Y^2 = 2^m X^4 + 3^n has rank 0 and torsion either Z/2 or (Z/2)^2
(happens when (m,n) = (0,0), (0,3), (1,0), (3,2), or (2, anything) mod 4).
So the only possible nontrivial solutions would occur in the four
cases when the rank is 1: (m,n) = (0,1), (0,2), (1,2), or (3,0) mod 4 .
So you would need to count the rational points on the 16 curves

Y^2= X^8 + 3 , Y^2= 16 X^8 + 3 , Y^2= X^8 + 243 , Y^2= 16 X^8 + 243,
Y^2= X^8 + 9 , Y^2= 16 X^8 + 9 , Y^2= X^8 + 729 , Y^2= 16 X^8 + 729,
Y^2= 2 X^8 + 9 , Y^2= 32 X^8 + 9 , Y^2= 2 X^8 + 729 , Y^2= 32 X^8 + 729,
Y^2= 8 X^8 + 1 , Y^2= 128 X^8 + 1 , Y^2= 8 X^8 + 81 , Y^2= 128 X^8 + 81

Each must have only a finite number of rational points, and as I say the
points are "probably" of modest height. So I had Magma search for all
rational points on all 16 curves, in each case checking for all X
of height up to 10 000 . In 14 cases all rational points found have
X = +1, −1, 0, or "1/0". The only exceptional cases are
Y^2 = 16 X^8 + 729 , with X = +− 3/2 and Y = +− 135/4
Y^2 = 32 X^8 + 729 , with X = +− 3 and Y = +− 459 .

(Since the corresponding 4 elliptic curves all have rank 1, one could
easily compute all the rational points on those elliptic curves out
to some great height, and check for X−coordinates which are perfect
squares. So it would not be much trouble to prove that any additional
points on these 16 hyperelliptic curves would have to have heights
greater than 10^6 or 10^9 or whatever.)

Assuming these lists of rational points are indeed complete, we
conclude that all the numbers of the form
2^m x^8 + 3^n z^8 (0 <= m,n <= 7 ; x, z >= 0)
which are square are those with:
x = 0 and n even
z = 0 and m even
x = z and (m,n) = (0,1), (3,0), or (4,2)
x = 3 z and (m,n) = (5, 6)
x = 3 u, z = 2 u, and (m,n) = (4, 6)

If x and z are to be powers of 2 and 3 respectively, this forces
x = z = 1 and (m,n) = (0,1), (3,0), or (4,2)

In particular, an integer of the form 2^m + 3^n is square iff
(m,n) = (0,1), (3,0), or (4,2).

dave
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