Re: First diagnal proof for real numbers

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WM wrote (in part):

> So we have now a fine final result concerning the
> first recognition of the special substitution rule
> necessary for Cantor's diagonal proof using n−ary
> representations of real numbers:
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> Borel (1898) probably was the first to mention it.

While working on something else (but related), I came
across two earlier appearances in the literature that
are explicit about this issue.

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Felix Klein, "Vorträge über ausgewählte Fragen der
Elementargeometrie", B. G. Teubner, 1895, v + 66 pages.
[Preface dated Ostern (Easter) 1895. JFM 26.0546.01]
http://quod.lib.umich.edu/cgi/b/bib/bibperm?q1=ACV2370.0001.001

Charlotte Angas Scott, Review of Felix Klein's "Vorträge
über ausgewählte Fragen der Elementargeometrie", Bulletin
of the American Mathematical Society (2) 2 (1895−96), 157−164.

The following is from pp. 162−163 of C. A. Scott's review,
which discusses what can be found on p. 42 of Klein's book.

The next step is to show how numbers can be
constructed that shall not be contained in this
orderly series. The number being required to lie
within certain limits, so that there are given
a certain number of decimal places, e.g., 5, the
digits in the following places have to be selected
so that the number differs from all of the series.
For a reason explained, the digit 9 is avoided.
The 6th digit is chosen to be different from the
6th of the first algebraic number, and thus the

number constructed will certainly be different
from this; the 7th digit is chosen to be different
from the 7th of the second algebraic number, by
which we ensure that the number written down is
not the second, and so on. Hence we are assured
of the existence of numbers that are not the roots
of any algebraic equation; that is, the existence
of transcendental numbers is proved, and it is shown
how they can be written down. Moreover, since the
choice of the digit to be written in any assigned
place is restricted only by the exclusion of two
digits, 9 and one other, we may choose any one
of the 8 that are left, zero being admissible in
this same way as any other. Hence between any two
algebraic numbers there are 8^oo transcendental
numbers, (not oo^8 as stated in the pamphlet,)
[the last 2 commas appear exactly as I've placed
them] and real algebraic numbers form only a small
part of all numbers.

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Heinrich Weber, "Lehrbuch der Algebra", Volume II,
Friedrich Vieweg und Sohn, 1896, xiv + 796 pages.
[Preface dated July 1896. JFM 27.0056.01]
http://dz1.gdz−cms.de/no_cache/dms/load/toc/?IDDOC=45274

Heinrich Weber, "Transcendental numbers", translation by
Wooster Woodruff Beman of Chapter 25 (pp. 745−767) of
Heinrich Weber's "Lehrbuch der Algebra" (Volume II),
Bulletin of the American Mathematical Society (2)
3 (1896−97), 174−195. [JFM 28.0084.01]

Pages 750−751 of Weber's book (Volume II) gives the
diagonal proof specifically for decimal expansions of
real numbers. There is a lengthy review of Weber's book
by James Pierpont in Bull. Amer. Math. Soc. (2) 4
(1897−98), 200−234, but as far as I could tell there
was no mention of either of Cantor's proofs in it.

The following is from pp. 178−179 of W. W. Beman's
translation:

This theorem may be demonstrated in another way
which is simpler in some respects and may be
briefly indicated. [The earlier proof was Cantor's
1874 proof.] We do not restrict the generality if
we confine ourselves to the interval from 0 to 1.
We shall imagine all numbers of this interval
represented by decimal fractions with an infinite
number of terms. Finite decimal fractions are

included if we make all the digits after a certain
one equal to zero. To render this representation by
decimal fractions unambiguous, it must be agreed
that for a finite decimal fraction this representation
must _always_ be chosen, so that, for example,
0.4999 ... must not be written for 0.5000 ...

We will now assume that these decimal fractions
form an enumerable mass. They may then be arranged
in a countable series, represented as follows:
[replace 'a' with '\alpha' and 'b' with '\beta']

\omega_1 = 0.a_{1}^{(1)}a_{2}^{(1)}a_{3}^{(1)} . . .
\omega_2 = 0.a_{1}^{(2)}a_{2}^{(2)}a_{3}^{(2)} . . .
\omega_3 = 0.a_{1}^{(3)}a_{2}^{(3)}a_{3}^{(3)} . . .

. . . .

where the a_{\mu}^{(\nu)} represent digits of the
decimal system.

But it is very easy to form a decimal fraction (or
indeed, as many as we please) which is not contained
in the series \Omega. We have only to form

\eta = 0.b_1b_2b_3 . . .

where the b_{\nu} are digits of the decimal system,
satisfying the one condition that for every \nu,
b_{\nu} is different from a_{\nu}^{(\nu)}. This
number \eta, which also belongs to the interval
(0,1) cannot be a number of the series \Omega.

The formation of \eta may be made still more
general by arbitrarily selecting the b's as far
as we please and _then_ applying the law,
b_{\nu} >< a_{\nu}^{(\nu)}.

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