Another Difficult Combinatorial Geometry Problem

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A set X in Rd is said to be m−convex , m. $(A!] (B 2 , iff for every m distinct points in X at least one of the line segments determined by those points lies in X. Ramsey $(A!/ (Bs theorem ,in graph theory, implies that the intersection of two 3−convex sets is 6−convex. For compact sets this is best possible in R4 . However in R2 the intersection of two compact , 3−convex sets is 5−convex. This note includes the best possible results for the , topology dependent, R2 case . The R3 case is the difficult problem of the Subject. These results complement those of the late H.G. Eggleston on vector sums of Valentine convex (aka 3−convex) sets.

Over fifty years ago ,Valentine [6] proved that a closed 3−convex set in the plane is decomposable into a union of 3 or fewer convex sets. In his concluding remarks he pointed out that the decomposition theory in R3 needed to be settled. That is still true today. In [7] he proved several important results about the points of local non−convexity of a closed 3−convex set. For such closed sets, points of local non−convexity lie in the kernel of the set and in the planar , closed case all points of local non−convexity are limit points of isolated points of local non−convexity.

Breen [1] produced topology dependent, decomposition theorems for planar 3−convex sets. The Introduction to [1] includes an explanation why a closed planar 3−convex set is simply connected.

Eggleston [3] constructed a remarkable compact 3−convex set in R4 which was not the union of finitely many convex sets. I believe that, at about the same time, Perles of the Hebrew University of Jerusalem independently produced a similar , but unpublished, example.

The broad results included in this note were mentioned in passing by Eggleston [4] including a reference to my 1978 PhD thesis, Calvert [2]. Eggleston was my great supervisor.

THEOREM Let X1 and X2 both be simply connected planar, 3−convex sets then X1 $(A!I (B X2 is 5−convex.

PROOF The outline of an elementary proof will follow two lemmas. The purely combinatorial and well known LEMMA 1 alone can be used to prove the theorem for several important special cases.

LEMMA 1. The only graph G of order 5 such that neither G nor its complement contains a triangle is C5 .

PROOF OF LEMMA 1. If some vertex of G has valency greater than or equal to 3 then clearly G or its complement contains a triangle. Hence G is C5.

REMARK 1 . An immediate consequence of LEMMA 1 is that if either X1 or X2 of the THEOREM is a union of two convex sets then the result follows. It is also easy to construct compact sets X1 and X2 with X1 having just one point of local non−convexity and X2 having just two points of local non−convexity with X1 $(A!I (B X2 not 4−convex see for example [2] page 48.

LEMMA 2 is also an immediate consequence of LEMMA 1.

LEMMA 2. If x1 $(A!− (Bx5 are five points of X1 $(A!I (BX2 with none of the associated line segments lying in X1 $(A!I (BX2 , in other words the x i are 5 visually independent points of the intersection, then no three of the xi are collinear.

PROOF OF THEOREM. The proof now follows easily but tediously from considering the convex hull of 5 supposedly visually independent points of the simply connected X1 and X2 . The cases where the convex hull is a triangle or pentagon are relatively straightforward whereas the case of a quadrilateral convex hull requires the consideration of the two cases : where two non−adjacent sides belong to each Xi or three belong to one Xi .

REMARK 2. It should be noted that my original detailed proof in [2] used a result of Erdos and Szekeres [5] that $(A!0 (Bfrom five points in the plane of which no three lie on the same straight line it is always possible to select four points determining a convex quadrilateral $(A!1 (B. However that proof required consideration of the location of the fifth point outside the convex hull of the other four.

REMARK 3. It is also noteworthy that X1 $(A!I (B X2 may fail to be 5−convex if just one of the Xi fails to be simply connected. One somewhat inelegant example of this is included in [2] pages 41and 42, where both Xi are also unions of three convex sets.

REMARK 4. Let xi 1 $(A!\ (B i $(A!\ (B 5 be five points on the moment curve with xi = p($(A&A (Bi) where 0 < $(A&A (Bi $(A!\ (B 1 and p($(A&A (B) = ($(A&A (B, $(A&A (B2, $(A&A (B3, $(A&A (B4) . As in [3] it is possible to construct compact X1 and X2 ,3−convex sets containing the xi with X1 $(A!I (BX2 not 5−convex.

REFERENCES

1. M.Breen, Decomposition theorems for 3−convex subsets of the plane, Pacific. J. Math. 53, (1974), 43−57.
2. D.I.Calvert, Generalisations of convexity, Doctoral Dissertation, University of London, (1978).
3. H.G.Eggleston, Valentine convexity in n dimensions, Math Proc. Camb. Phil. Soc., 80 (1976), 223−228.
4. H.G.Eggleston, Vector sums of Valentine convex sets, Math. Proc. Camb. Phil. Soc., 92 (1982), 17−19.
5. P.Erdos and G.Szekeres, A combinatorial problem in geometry, Compos. Math. 2 (1935), 463−470.
6. F.A.Valentine, A three point convexity property, Pacific J. Math. 7 (1957), 1227−1235.
7. F.A.Valentine, The intersection of two convex surfaces and property P3, Proc. Amer. Math. Soc. 9 (1958), 47−54.

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