Root Finder and Example

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DERIVING ALL ROOTS TO ANY POLYNOMIAL (with example)

Let
N=(a[1],a[2],...,a[n])
T=(t,t^2,t^3,...,t^n)
then
N*T+a[0]=0 = nth degree polynomial in t. N=(a[1],a[2],a[3],...,a[n]) is the normal to the plane and T is a space curve in t. Define Q=(−a[0]/|N|^2)N and let B,P be 2 points on the plane such that (Q−B)*(P−B)/(|Q−B||P−B|) does not equal 1 or −1. (B,P are selected so that (Q−B) and (P−B) are not parallel). The unit vector R on the plane perpendicular to Q−B is,

"P−B minus the reflection of P−B onto the unit vector of Q−B, divided by the magnitude of the same."

$$R = \frac{P-B - \dfrac{(P-B)*(Q-B)}{|Q-B|^2}(Q-B)}{\left| P-B - \dfrac{(P-B)*(Q-B)}{|Q-B|^2}(Q-B) \right|}$$

also define,

$$S = \frac{Q-B}{|Q-B|}$$

then

T = B + uR + mS where u is some ratio and m is some ratio. Taking the dot product of S with both sides and of R with both sides, note that R*S=0 and

m=(T−B)*S= S*T−B*S
u=(T−B)*R= R*T−B*R

T = B + (R*T−B*R)R + (S*T−B*S)S
T' = (R*T')R + (S*T')S
T'' = (R*T'')R + (S*T'')S
.
.
T^{n}=(R*T^{n})R + (S*T^{n})S where T^{n} is the nth derivative of T
with respect to t. For t=0,

T(0)=(0,0,0,..,0)
T'(0)=(1,0,0,...,0)
T''(0)=(0,2,0,...,0)
.
.
T^{n}(0)=(0,0,0,...,0,n!)

If R=(r[1],r[2],r[3],...,r[n]) and S=(s[1],s[2],s[3],...,s[n]),

T(0)=B−(B*R)R−(B*S)S
T'(0)= r[1]R + s[1]S
T''(0)=2r[2]R+2s[2]S
.
.
T^{n}(0)=n!r[n]R+n!s[2]S

         n
T(t) = SUM[(1/p!)T^{p}(0)t^p]
        p=0

Where T^{p}(0) is t=0 in the pth derivative of T (Maclaurin) Otherwise,

T(t) =
      B−(B*R)R−(B*S)S +
     (r[1]R+s[1]S)t +
     (r[2]R+s[2]S)t^2 +
     (r[3]R+s[3]S)t^3 +
             .
             .
     (r[n]R+s[n]S)t^n Let B=(b[1],b[2],...,b[n])

Decoding T(t),

(r[1]r[1]+s[1]s[1]−1)t +
(r[2]r[1]+s[2]s[1])t^2+
(r[3]r[1]+s[3]s[1])t^3+
.
.
(r[n]r[1]+s[n]s[1])t^n + (b[1]−(B*R)r[1]−(B*S)s[1]) = 0
==
(r[1]r[2]+s[1]s[2])t +
(r[2]r[2]+s[2]s[2]−1)t^2+
(r[3]r[2]+s[3]s[2])t^3+

.
.
(r[n]r[2]+s[n]s[2])t^n + (b[2]−(B*R)r[2]−(B*S)s[2]) = 0
==
(r[1]r[3]+s[1]s[3])t +
(r[2]r[3]+s[2]s[3])t^2+
(r[3]r[3]+s[3]s[3]−1)t^3+
.
.
(r[n]r[3]+s[n]s[3])t^n + (b[3]−(B*R)r[3]−(B*S)s[3]) = 0
==
This continues until we arrive at,
(r[1]r[n]+s[1]s[n])t +
(r[2]r[n]+s[2]s[n])t^2+
(r[3]r[n]+s[3]s[n])t^3+
.
.
(r[n]r[n]+s[n]s[n]−1)t^n + (b[n]−(B*R)r[n]−(B*S)s[n]) = 0

Generalizing n equations into n planes,

(r[1]r[1]+s[1]s[1]−1)x[1]+
(r[2]r[1]+s[2]s[1])x[2]+
(r[3]r[1]+s[3]s[1])x[3]+
.
. D R14 S14
(r[n]r[1]+s[n]s[1])x[n] + (b[1]−(B*R)r[1]−(B*S)s[1]) = 0
==
(r[1]r[2]+s[1]s[2])x[1]+
(r[2]r[2]+s[2]s[2]−1)x[2]+
(r[3]r[2]+s[3]s[2])x[3]+
.
.
(r[n]r[2]+s[n]s[2])x[n] + (b[2]−(B*R)r[2]−(B*S)s[2]) = 0
==
(r[1]r[3]+s[1]s[3])x[1]+
(r[2]r[3]+s[2]s[3])x[2]+
(r[3]r[3]+s[3]s[3]−1)x[3]+
.
.
(r[n]r[3]+s[n]s[3])x[n] + (b[3]−(B*R)r[3]−(B*S)s[3]) = 0
==
This continues until we arrive at,
(r[1]r[n]+s[1]s[n])x[1]
(r[2]r[n]+s[2]s[n])x[2]+
(r[3]r[n]+s[3]s[n])x[3]+
.
.
(r[n]r[n]+s[n]s[n]−1)x[n] + (b[n]−(B*R)r[n]−(B*S)s[n]) = 0

These n planes intersect each other with various lines (they can all be found), they have angles between all of their normals (also easily obtained), and share a common point (the n equations can be solved for the n unknowns using linear algebra). Supposing this common point is calculated as C=(c[1],c[2],c[3],..,c[n]). It can be deduced that while the point C is a solution to the intersection of all the planes, it does not necessarily follow that c[2]=c[1]^2, c[3]=c[1]^3, etc as in T=(t,t^2,t^3,..,t^n). Supposing the solution t=w exists. It can be inferred that if w is a root, it must also satisfy the point of intersection of the planes at C. Supposing, then, that C is a solution.

Selecting the last component c[n]=t^n,

t = c[n]^(1/n)[cos(2k*pi/n)+ i sin(2k*pi/n)] if c[n] is positive

t = |c[n]|^(1/n)[cos((2k+1)pi/n)+ i sin((2k+1)pi/n)] if c[n] is negative

where k = 0,1,2,3,...,n−1 Binomial Equation

Rendering the n roots for the nth degree polynomial N*T+a[0]=0.

EXAMPLE

Solve Bx/(2A)=sin(x/2) for B/A = 2*(2^*1/2)/pi

x = pi/2

B=chord
A=arclength
x=angle subtending arc and chord

sin(x/2) is expanded as a power series, and x/2 is canceled from both sides. Let t=(x/2)^2 and carry the series out to the 8th power of t. The resulting values were obtained using the above method calculated on a spreadsheet:

a[j] x[j]=t^j c*pi
a[0] 0.099683684
a[1] −0.166666667 −6.14244E−12
a[2] 0.008333333 −11.53827506
a[3] −0.000198413 18.70774654 0.845001123
a[4] 2.20459E−05 8.206996951 0.538761056
a[5] −2.50521E−08 18.71340883 0.571848134
a[6] 1.6059E−10 8.206367733 0.452073056
a[7] −7.64716E−13 18.71340954 0.483711329
a[8] 2.81146E−15 8.206367728 0.414113129

B P Q
0.645853126 2 0.596609723
1 28.05647943 −0.029830486
2 1 0.00071025
1 2 −7.89166E−05

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2 1 8.9678E−08
1 2 −5.74859E−10
2 1 2.73742E−12
1 2 −1.00641E−14
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DISCUSSION

All the values in the c*pi column should be equal to 1/2. The values obtained are somewhat of a disappointment, especially after carrying out the power series to the 8th power of t (the 8th term is on the order of 1/17!). The negative values in the x[j]=t^j column give complex solutions, as do each t^j decomposed with the Binomial Equations.

The B and P that I selected were designed so that P−B and Q−B are not parallel (they are close to pi/2 apart). However, the 28.05647943 term may have been too large and introduced error.

Nonetheless, it is somewhat heartening that the c*pi column values are somewhat close to 1/2, showing that within some range of certainty, the method pans out the right answers.

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