Re: Symmetric rational approximations of the surface area of an ellipsoid

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David W. Cantrell <DWCantrell@sigmaxi.org> wrote in message
news:<20040529195723.225$tP@newsreader.com>...
> [This post is to supersede my original one, in which the last exponent in
> the numerator of (1) was erroneously typed as being 3. It should be 2, as
> now shown below. Sorry for the typo!]
>
> A remarkable (new?) approximation for the surface area of an ellipsoid is
> presented, followed by miscellaneous comments.
>
> Let a, b, and c denote the lengths of the semiaxes of an ellipsoid, and let
> P, Q, and R denote the elementary symmetric polynomials
>
> P = a+b+c, Q = ab+bc+ca, and R = abc.
>
> Then the surface area of the ellipsoid may be approximated,
> with |relative error| < 0.0013, by
>
> [Please view in a fixed−width font.]
>
> 15 Q^3 − 7 PQR − 27 R^2
> 4pi −−−−−−−−−−−−−−−−−−−−−−−− (1)
> 15(2 Q^2 + PR)
>
> This symmetric rational approximation is notable for its combination of
> simplicity and accuracy. I wonder if it is new. [My copy of "The surface
> area of an ellipsoid" by P. A. P. Moran (pp. 511−518 in _Statistics and
> Probability: Essays in Honor of C. R. Rao_, eds. G. Kallianpur, P. R.
> Krishnah and J. K. Gosh, North−Holland, Amsterdam, 1982) has, alas, been
> "temporarily misplaced". That's the only reference which comes to mind
> which I think might have already given this approximation.]

I've now had the opportunity to look again at the article by Moran.
(1) is not given there. I must suppose that it is a new approximation.

> The approximation gives the surface area precisely both in the degenerate
> case, when an axis is of length zero, and in the spherical case.

[snip]

> One such approximation is
>
> 5 P^2 R + 4 Q R − 6 P Q^2
> 4pi −−−−−−−−−−−−−−−−−−−−−−−−−− (4)
> 3(R − 4 P Q)
>
> which, between the extreme cases, overestimates the surface area, with
> relative error < 0.011. Of course, this is not impressive compared to (1),
> which is much more accurate and almost as simple. I must note that I do not
> know if (4) and (1) are "aesthetically optimal" extreme−perfect
> approximations of types [5,3] and [6,4] resp. Certainly they are not
> optimal if one's only desire is to minimize max|relative error|. Numerical
> techniques could be used to determine _many_messy_ coefficients so that an
> extreme−perfect approximation of a certain type would minimize max|relative
> error|. But if, as I suspect, the maxima of |relative error| would not be
> far smaller than those of (4) and (1), then I would not call such messy
> approximations "aesthetically optimal". Anyway, as I said, there may, for
> all I know, be symmetric extreme−perfect rational approximations which are
> "nicer" than (4) and (1).

Consider surface area approximations of the form

$$2\pi \, \frac{Q^3 - d\,PQR - (9-9d-6e)\,R^2}{Q^2 + e\,PR} \qquad (6)$$

It is easily verified that, regardless of the values of d and e, they are extreme−perfect, that is, they give the surface area precisely in the extreme cases (when the ellipsoid is a sphere and when it is degenerate, having an axis of length zero). Note that this form is a generalization of (1), which may be obtained using d = 7/15 and e = 1/2 in (6).

Determining d and e numerically so that the maximum of |relative error| is minimized, we find that d = 0.44882... and e = 0.51789... , and the surface area is then approximated by (6) with |relative error| < 0.0008. This supports my previously stated suspicion that the worst |relative error| would "not be far smaller" than 0.0013, as provided by (1). But of course, it is smaller nonetheless, and so people wishing specifically to minimize worst |relative error| may prefer to use (6) with the numerically determined values of d and e. I still tend to prefer (1) normally: It is very simple and, for expressions in form (6), it is optimal for nearly spherical ellipsoids.

Very shortly after posting my original article, Knud Thomsen sent me a related approximation for the surface area. His suggestion, equivalent to taking d = 9/20 and e = 31/60 in (6), can be presented neatly as

$$6\pi \frac{20 Q^3 - 9 PQR - 37 R^2}{60 Q^2 + 31 PR} \quad (7)$$

and provides |relative error| < 0.00082. Thus his approximation is admirable in that, while using fairly small integer coefficients, it manages to provide a bound on |relative error| which is quite close to the smallest obtainable using form (6).

David W. Cantrell