Re: Lattices−−the distributive inequality

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From: William Elliot (*marsh_at_privacy.net*)
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From: Van Jacques <calccurve−test23@yahoo.com>
Newsgroups: sci.math
Subject: Lattices−−the distributive inequality

>Consider lattices L as posets with meet(a,b) == a /\ b == glb(a,b),
>and join(a,b) == a \/ b == lub(a,b), for any a,b in L.

>L.E. means "less than or equal to".

>(*) Prove that (a /\ b) \/ (a /\ c) L.E. a /\ (b \/ c)

the same in simpler Boolean algebra like notation
    ab + ac <= a(b + c)
which I use unless notational conflict arises.

>With an equal sign, this is the distibutive law for distributive
>lattices.
Both
    a(b + c) = ab + ac
and
    a + bc = (a + b)(a + c)
is the distributive law for lattices

>We also have the dual to (*), obtained by interchanging /\ with \/,
>and changing L.E. to G.E.

a + bc <= (a + b)(a + c)

>Here are some examples of lattices:

>1) the power set 2^X == P(X) of X ordered by set−theoretic
>inclusion, including Boolean algebras.

>2) the integers Z with L.E. the usual order,
Exercise: show any linear or total order is distributive
lattice by glb = min, lub = max

>3) the positive integers ordered by divisibility, or dual to this
>4) the ideals of Z ordered by set−theoretic inclusion,

>5) the subgroups of a groups ordered by set−theoretic inclusion
>6) the intermediate fields of a extension E of a field F, again
> ordered by set−theoretic inclusion.

>I think all of these are distibutive, though
>though I haven't checked it except for P(X).

Check the subgroups of Z_2 x Z_2
{ (0,0) }
{ (0,0), (1,0) }
{ (0,0), (0,1) }
{ (0,0), (1,1) }
{ (0,0), (0,1), (1,0), (1,1) }
cf, second diagram example below

Also the lattice of topologies for a set with more than three points isn't distributive.

>Also, a L.E. b <==> a \/ b = b <==> a /\ b = a
>follows from the definitions.
Yes, yet it depends upon how you define a lattice.
Tho ultimately do not all theorems follow from the definitions?

You can define a lattice simply from the order and the requirement that lub x,y and glb x,y exist for all x,y.

Or you can define a lattice axiomatically
(S,*,+) is lattice when SS, S+S subset S,
x(yz) = (xy)z, xy = yx
x+(y+z) = (x+y)+z, x+y = y+x,

ie S is closed under associative and commutative operators * and +
with the added axiom, the order axiom
a+ab = a = a(a+b)

Then you define a <= b as a = ab and show
<= is an order and that
ab = glb a,b; a + b = lub a,b

Prior to doing that you'll want to show
aa = a = a+a; a = ab iff a + b = b
Just for fun, here's an extra for you
ab = a + b ==> a = b

Conversely from the order <= and the existence of glb a,b and lub a,b and by notating ab = a*b = glb a,b and a+b = lub a,b, derive all the axioms for a lattice.

>I thought perhaps this could be used to show that
>(*) is true, but I still don't see it.
b,c <= b + c; ab, ac <= a(b +c); ab + ac <= a(b + c)

I leave for you to show
    a + bc <= (a + b)(a + c)

>I see that the distributive law is true, but I
No! Not all lattices are distributive, for (monospace!) example

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  1 1
 / \ /|\
u \ u v w
| w \|/
v / 0
 \ /
  0
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>don't see how to prove the above inequality (*).
In a distributive lattice the inequality is trivial.

A classic reference is Davey, B. A.,
Introduction to Lattices and Order

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