Re: normalizer of Q8 in SL(2,q)

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Edwin Clark wrote:
>
> *A friend is interested in the following question.*
> *Any help would be appreciated.*
>
> *Let q be an odd prime power and let N be the normalizer of Q8 in SL(2,q)*
> *where Q8 is the subgroup generated by the two matrices*
>
> *[0 1] [a b]*
> *[−1 0] , [b −a]*
>
> *where a^2 + b^2 = −1.*
>
> *Is N/Q8 always either cyclic of order 3 or isomorphic to S_3?*
>
> *Edwin Clark*

Robin Chapman gave an excellent answer that depends on the knowledge of subgroups of PSL(2,q) of order prime to q. Me the mortal takes a more computational view: Assume that

[u_11 u_12]
[u_21 u_22]

centralizes Q8. This commutes with the first generator, iff u_11=u_22=v, and u_12=−u_21=v. Using this information we further see that this matrix commutes with the second generator, iff bv=av=0. So we must have v=0, and hence u= \pm 1.

Therefore the centralizer of Q8 in SL(2,q) is of order 2 and is contained in Q8 itself. The group Q8 has 24 automorphisms of which 4 are inner. It is also well known that the quotient group of the automorphism group by the inner automorphisms is isomorphic to S_3. Hence N/Q_8 is isomorphic to a subgroup of S_3.

To prove that # N/Q_8 is divisible by 3 it suffices to show that there exists a matrix X in SL(2,q) such that conjugating the first

generator by X gives you the second. This does seem to be more
difficult. By looking at the eigenspaces (corresponding to eigenvalues
that are primitive fourth roots of unity) it is easy to see that
both of the generators are similar to the same diagonal matrix.
Alas, to be sure that the fourth roots of unity exist, you may need
to go to SL(2,q^2) instead. If I think of a way around that obstacle,
I will come back to this thread.

Cheers,

Jyrki Lahtonen, Turku, Finland