Re: 1+i > i

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From: mina_world <mina_world@hanmail.net>
Newsgroups: sci.math
Subject: 1+i > i

>(1+i)−i = 1 > 0
>=> (1+i) > i

>um........of course, impossible ??
No, not at all. You can extend the order of R to C in a number of ways.
However, since C is a group and a ring, you'd want to extend the order
of R to C in such a way that C is an ordered group and ordered ring.

You could give C the (partial) order
    a + bi <= x + yi when a <= x, b <= y.
Prove this is an order and that it's translation invariant, ie
    a + bi <= x + yi ==> a+bi + u+vi <= x+yi + u+vi
This makes C an ordered group. However with that order, C is not
an ordered ring because the ordered ring axiom
    0 <= a + bi, x + yi ==> 0 <= (a + bi)(x + yi)
is not true for all a,b, x,y.

(The ordered ring axiom
    0 <= r,s ==> 0 <= rs
is equivalent to the more useful
    0 <= t, r <= s ==> tr <= ts.)

Now fitting in with your observation 1 + i > i
    a + bi <= x + yi when a <= x, b = y
is an (partial) order for C, which is easy to see.
Also easy to verify is that <= is translation invariant, hence
(C,<=,+) is an ordered group and that the ordered ring axiom for
<= holds, thus showing (C,<=,+,*) is an ordered ring, with the
very (partial) order you have noticed.
    You will also find that this order is Dedekind complete, bounded
complete or complete within bounds, ie that bounded sets have inf and sup.
Notice, that elements of a bounded set will all have the same imaginary
component. Thus this order isn't directed.
    A vizualation of this order is for each y, { x + yi | x in R }

is a copy of totally ordered R and that elements a + bi, x + yi with
differing imaginary components are incomparable, ie
        not (a + bi <= x + yi or x + yi <= a + bi)

>i know that complex numbers can't compare about size.
>but this process is very plausible as paradox.
Plausible, yes. Paradox, no. Useful, little as it's mostly ignored.

Exercise: C can not be a totally ordered ring.
Exercise: C is a totally ordered group with lexicographical order
        a + bi <= x + yi when a < x or a = x, b <= y
However you will find C with that order is not bounded complete as
for example, { yi | y in R } which is bounded above by 1, has no sup.

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Problem:  show the only order ring extension of ordered R to C is
        a + bi <= x + yi when a <= x, b = y
This is equivalent to showing that the only elements
c of C, for which 0 <= c, are the non-negative reals.
So assume
        0 <= a + bi, b /= 0
Now if a <= 0, then
        0 <= -a <= bi;  0 <= (bi)^2 = -b^2;  b = 0
Thus 0 < a and
        if a^2 <= b^2
                0 <= (a + bi)^2 = a^2 - b^2 + 2abi
                0 <= b^2 - a^2 <= 2abi
                0 <= -4a^2 b^2, not so
So b^2 < a^2;  -a < b < a
Again
        0 <= a^2 - b^2 + 2abi
        0 < b^2 <= a^2 + 2abi
        0 <= a + 2bi
        0 <= a/2 + bi
Hence from the previous
        -a/2 < b < a/2
Continuing this way
        -a/4 < b < a/4
        ...
Thus eventually b = 0.
Is there an easier way of showing this?
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