Re: how to prove that f^2+f ' ^2 <=1 if ...

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In news:<200408191114.i7JBEjf18741@proapp.mathforum.org>
schrieb eric detourre <eric.detourre@laposte.net>:
> *Here is the following problem :*
>
> *Prove that: f^2 + f ' ^2 <= 1*
> *when f is twice differentiable over R*
> *and f^2 <= 1*
> *and f ' ^2 + f " ^2 <= 1 (1 is not less than the sum*
> *of the square of the first and second derivative of f).*
>
> *I don't have the slightest idea on how to assert this, and this result*
> *(which happened to be an exercise in a french engineering school)*
> *may be false (but i found no counter−example)*
> *as it is false when replacing R with any subset of R.*

The result is correct,
but my proof is rather longish.
It is based on the usual procedures to solve
the ordinary differential equation
g' ^2 + g" ^2 = 1,
but you have to be very careful!

I welcome any suggestions for improvement!

So, we want to show

f(x)^2 + f'(x)^2 <= 1

for all points x in R.

In case that x is a critical point of f, i.e. f'(x) = 0,
the claim follows from f(x)^2 <= 1.

So let x now be a regular point of f:

x in U := {y in R | f'(y)<>0 }

U is an open subset of R, hence a disjoint union of open, non−empty
Intervals I_i.
For every such interval, the function f is a stronly monotonic function
on the closure cl(I_i), thus bijective,
and we can write the derivative f' as a function F_i of f on cl(I_i):

f' = F(f)

(I dropped the index − the interval will now be called I, the function F).
Then
f'' = F'(f) * F(f),

and the inequality f' ^2 + f'' ^2 <= 1 yields

F(f)^2 + F'(f)^2 * F(f)^2 <= 1

F'(f)^2 * F(f)^2 <= 1 − F(f)^2

The next step will only work on intervals I on which
f' does not take on the values +/− 1. −−
Division by (1 − F(f)^2) >(!!) 0:

| F(f) * F'(f) / sqrt(1 − F(f)^2) | <= 1
| d/df [sqrt(1 − F(f)^2)] | <= 1 (*)

Now comes the part, where we have to use that f is defined and bounded
on *all* of R [at least, we need to use it when I is an unbounded interval]:

Given I=I_i and a fixed point x in I,
there are for every epsilon > 0
points x− = x−(epsilon) and x+ = x+(epsilon) such that

x− in I, x+ in I, x− <= x <= x+
and
f'(x−)^2 <= epsilon and f'(x+)^2 <= epsilon

[If I = (a,b) is finite from from below, a <> −oo,
existence of x− follows from f'(a)=0,
in case a = −oo, existence of x− follows
from the fundamental theorem of differential calculus,
applied to f on the interval [x−2/sqrt(epsilon),x]:

f'(x−) = (f(x) − f(x−2/sqrt(epsilon)))/(2/sqrt(epsilon))

−−> |f'(x−)| <= 2/(2/sqrt(epsilon)) = sqrt(epsilon)

Similarly for x+.]

Now, integrate Inequality (*)
over the interval [f(x−),f(x)] or [f(x+),f(x)],
depending on whether f is monotonically de− or increasing.

Let's take the case that f is decreasing.
Let's also say, that epsilon has been taken to be smaller than F = f'(x)^2.
Then:

sqrt(1 − epsilon) − sqrt(1 − F(f(x))^2) <= f(x−) − f(x)
<= 1 − f(x)

sqrt(1 − F(f(x))^2) >= sqrt(1 − epsilon) − 1 + f(x)

The same result follows if f is increasing on I from integration
over [f(x+),f(x)]. Now take the limit epsilon to 0:

sqrt(1 − F(f(x))^2) >= f(x)
i.e.:
sqrt(1 − f'(x)^2) >= f(x)

That is one half of the desired inequality.
For the other half, sqrt(1 − f'(x)^2) >= −f(x),
you will have to start again at (*), but this time integrate it
over [f(x+),f(x)] if f is decreasing on I, resp.
over [f(x−),f(x)] if f is increasing:
Example:

sqrt(1 − epsilon) − sqrt(1 − F(f(x))^2) <= f(x) − f(x−)
<= −(−f(x)) − 1

sqrt(1 − F(f(x))^2) >= sqrt(1 − epsilon) − 1 + (−f(x))

Here ends the proof of
f(x)^2 + f'(x)^2 <= 1
for x in connected components I = I_i
of the set of regular values of f,
on which f' does not take on either of the values 1 and −1.

If I = I_i is an open interval on which f' becomes 1 or −1,
do the following:
Since I is an interval of regular values of f,
we may w.l.o.g. assume that f'>0 on I,
and thus that f' takes on the value +1 somewhere on I.
[The other case is analogous.]

The remaining open set
I minus {y in I | f'(y) = 1}

decomposes again into a disjoint union of non−empty open intervals J_j.
We will show that {y in I | f'(y) = 1} can only consist of a
single point, that at this point f is zero,
and that in conclusion f will be a sine function (up to translation).
>From this,
f^2 + f' ^2 = 1
follows automatically.

So, let x be a point in I with f'(x) = 1.
Fix a positive constant epsilon > 0.
Then there will again be points x− < x < x+ in I
such that f'(x−) = epsilon and f'(x+) = epsilon.
Let's say, x− lies in J_1, and x+ lies in J_2, J_1 disjoint from J_2.
Then, take the inequality
    f' ^2 + f'' ^2 <= 1,

say on J_1, and rewrite it:
    | f'' / sqrt(1−f' ^2) | <= 1

[Recall that f' ^2 <> 1 on J_1!]

    | d/dx (arcsin(f')) | <= 1

Integrate over an interval [z,sup(J_1))
with z in J_1 intersected with [x−,x]:

    −−> arcsin(1) − arcsin(f'(z)) <= b − z with b:= sup(J_1)

    f'(z) >= sin(pi/2 − (b−z)) as long as 0 < (b−z) < pi/2 (**)

But since x− is a point, where f' takes a value very near to zero,
it follows from (**) that
    f'(z) >= sin(pi/2 − (b−z)) *and* 0 < (b−z) < pi/2
holds [up to order O(epsilon)] for *all* points
in the interval (a,b) := (x−,x) intersected J_1.

Thus the difference f(b) − f(a)
will be

    f(b) − f(a) = int_a^b f'(z) dz
>= int_(b−pi/2)^b sin(pi/2 − (b−z)) dz
        = 1 − 0

    −−> f(x) >= f(b) [since x >= b]
>= f(a) + 1
>= −1 + 1

Of course, there are still some terms of order O(epsilon) in the last
inequality, but at this point epsilon can go to zero,
and we obtain
    f(x) >= 0.

Analogously, but argueing over the interval (x,+x) intersect J2,
one obtains

    f(x) <= 0,

thus f(x) = 0,
and in addition, all inequalities met inbetween "must" be equalities,

so in particular, the complement of {f'=1} in I
must not contain any other components than J_1 and J_2,
and {f'=1} itself must be of measure zero.
Thus

I = J_1 u {x} u J_2,

and

f must be identical to z |−> sin(z−x) on I.

This ends the proof.

PFFFF.... −− I'm exhausted!
Sorry, but I don't have the nerve to proofread the whole text right now.
Maybe later...

Thomas