classifying degenerate critical points

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Hi,

I'm looking for some help in classifying degenerate critical points as minima, maxima, or saddles. Assuming a function f : R^n −−> R, the critical points are those where the gradient of f is zero. The "standard" technique for classifying critical points involves examination of the eigenvalues of the Hessian at the critical points. This is basically an extension of the Second Derivative Test to the multivariate case, and is a consequence of Taylor's theorem. However, when the Hessian is not invertible (i.e. when the critical point is "degenerate" or "non−Morse"), this test breaks down, much as the Second Derivative Test breaks down for, say, f(x)=x^4. For this scalar example, we have to examine higher order derivatives to classify the critical points. Is there a similar technique for the multivariate case?

Here are three examples I've been struggling with, all of which have degenerate critical points at (0,0):

f1(x,y) = x^4 + y^4
f2(x,y) = x^3−3xy^2
f3(x,y) = (x−y^2)(x−2y^2)

The first is clearly a maximum, the second is the so−called monkey saddle, and the third is also a saddle. Historical note: f3 was the counter−example used by Peano to disprove a proposition by Lagrange −− see pg 33 of Hancock's 1917 text "Theory of Maxima and Minima", available free online at this long url −−
http://www.hti.umich.edu/cgi/t/text/text−idx?c=umhistmath;idno=acl8264.0001.001

The books I've looked in all seem to just gloss over the issue of degenerate critical points, and they state something like, "if the Hessian is singular, then the test is indeterminate. The End." So, any help in finding a technique to classify degenerate critical points would be great, and a reference on this topic would also be wonderful. I'm not even sure what topic this would fall under, or where to begin looking.

Thanks in advance. . .
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