

Re: Uncountable sets in CZF?

Source: <http://sci.tech-archive.net/Archive/sci.math/2004-09/1404.html>

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Date: 09/07/04

Date: 07 Sep 2004 16:16:40 GMT

In article <cgkfng\$abo\$1@bunyip.cc.uq.edu.au>, D.McAnally@i'm_a_gnu.uq.net.au

(David McAnally) writes:

|The Platonic attitude that there is exactly one set \mathbb{R} of real numbers is
|not fully appropriate.

I disagree. Even while wearing my constructivist hat I still disagree.

If you're going to adopt the attitude that there isn't one, which is up to you, you still are left with the fact that as far as ZF is concerned, there exists a unique set \mathbb{R} of real numbers. So if you're in the middle of doing some reasoning inside ZF, your reasoning should respect that. If you are not in the middle of doing some reasoning based on ZF, then a few words about what you are doing instead might be in order.

I think trying to think of the reals as not being unique can easily lead to confusion if you don't carefully partition such beliefs off from the mathematics one is trying to do, if it's in a system that regards the reals as being unique. If there's some alternative metatheory in which the unique existence of \mathbb{R} is not a theorem, but the results from model theory you want to use are, it should be named.

|One cannot assume that there is exactly one set \mathbb{R}
|that must be common to all models of ZF,

Of course.

|or that models of ZF are flawed
|on the basis that \mathbb{R} can be altered by taking a generic extension.

It depends on what you mean by "flawed".

Certainly it's possible for a model of ZF to have the real line as its \mathbb{R} . (If you like, think of it as a theorem of ZF. ZF doesn't have a theorem saying there are standard models of ZF, but adding an assumption such as the existence of a measurable cardinal, then there

exists a model M of ZF where the set of reals relative to M is \mathbb{R} .)
Such a model might be considered more perfect.

In article <cgfu52\$uev\$1@bunyip.cc.uq.edu.au>, D.McAnally@i'm_a_gnu.uq.net.au
(David McAnally) writes:

|For myself, the easiest way to understand generic extensions is through
|Boolean-valued models.

Nice of you to go to the trouble of explaining that in your posting.
It looks like you tersely covered a good fraction of the key points of
the development of boolean-valued model as it's done in textbooks.

A Boolean-valued model M^G can be arranged to behave the way you
describe; we can wind up with an element f in M^G such that
| $f:Z^{\wedge} \rightarrow R^{\wedge}$ is a bijection| = 1, where Z^{\wedge} and R^{\wedge} are the counterparts
of Z and R in M^G , although R^{\wedge} is not what M^G considers to be \mathbb{R} . This
 f is essentially a function from $Z \times R$ to G . The most obvious choice
of f in this particular case is symmetrical under permutations of Z
and permutations of R (which both act on G).

The model can get away with thinking of "for each n , there exists an
 r such that $f(n)=r$ " as having a truth-value of 1 because the value of
"there exists an r such that $f(n)=r$ " is the least upper bound of the
values of " $f(n)=r$ " ranging over all r in \mathbb{R} .

I think the problem arises when you try to apply the equivalence of
the Boolean-valued approach (which doesn't need to construct an actual
model of ZF) with a forcing approach that does.

In article <cgfun1\$197\$1@bunyip.cc.uq.edu.au>, D.McAnally@i'm_a_gnu.uq.net.au
(David McAnally) writes:

|A set G (not necessarily an element of V) is called a generic set if

- |
- | (1) G is a subset of F (i.e. all the elements of G are forcing
| conditions),
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- | (2) if p is an element of G , then any condition q weaker than p
| is an element of G ,
- |
- | (3) if p and q are elements of G , then p and q are compatible,
|
- | (4) if D is an element of V and D is a dense set of conditions,
| then G and D have nonempty intersection, i.e. G intersects
| every dense subset of F .
- |

|For all x in V^B , define $i_G(x)$ by induction on x by

| $i_G(x) = \{ i_G(y) : y \text{ in } \text{dom}(x), G \text{ intersects } x(y) \}$,

| and define $V[G]$ to be the class with elements $i_G(x)$ for x in V^B .

I wish you hadn't chosen to use the letter "V" for your model, since V is the usual notation for the class of all sets. It requires a little care to apply the construction to a proper class, but it can be done. One useful trick is to realize that when we're working with proper classes, we're really dealing with the predicates that define them. So with V, really we're working with the predicate that's always true, like $0=0$.

If we apply this process to V, V^G can be a boolean-valued model that fails to satisfy various statements true in V. If we reduce back to a 0-1-valued model using such a mapping i_G , though, we go right back to V. It doesn't leave us with any new elements than what we started with.

In what sense are none of the elements new? In the sense that they are in V, i.e. that $0=0$. That's just a matter of definition. It's not a metaphysical claim about "the real world of sets" or anything. The whole discussion still can be regarded as an informal version of an argument to be formalized in the language of ZF.

[Then $V[G]$ is the smallest model of ZFC which contains V as a submodel and contains G as an element. $V[G]$ has the same ordinals as V, and the same constructible sets as V.

In the case where we use V to mean the class of all sets, G already was an element of V.

[For a formula $\phi(x_1, \dots, x_k)$ with free variables x_1, \dots, x_k , and for elements y_1, \dots, y_k , of V^B , $V[G]$ satisfies $\phi(i_G(y_1), \dots, i_G(y_k))$ iff there exists a condition p in G such that p forces $\phi(y_1, \dots, y_k)$.

In our main example, all conditions force f to be a bijection, but $i_G(\mathbb{R}^\wedge) = \mathbb{R}$ and $i_G(\mathbb{Z}^\wedge) = \mathbb{Z}$, and there is no bijection between \mathbb{Z} and \mathbb{R} .

I believe there is a problem with preserving quantifiers. When we are evaluating a formula with a quantifier in V^B , we take the least upper bound (for existential quantifiers) or the greatest lower bound (for universal quantifiers). For this to be preserved under reduction to a 0-1-valued model, it needs to be the case that when each member of the set of elements of B whose greatest lower bound we are taking is in the filter, then so is their greatest lower bound.

[Define G' in V^B by $\text{dom}(G') = \{ \hat{p} : p \in F \}$, and $G'(p) = e(p)$ for all conditions p in F, then $\|G'\| = 1$, and $i_G(G') = G$. It follows that if ZFC is consistent, then the existence of a generic set is consistent with ZFC.

[The connection between generic ultrafilters and generic sets is given as follows.

[If G is a generic set, then

|
| $U = \{ A \text{ a regular open set in } F : A \text{ intersects } G \}$
|
| is the corresponding generic ultrafilter, and $V[U] = V[G]$.
|
| If U is a generic ultrafilter, then
|
| $G = \{ p \text{ in } F : e(p) \text{ in } U \}$
|
| is the corresponding generic set, and $V[G] = V[U]$.

I think it's only the case that we get back a 0–1–valued model of ZF satisfying the sentences whose values in V^B lie in U , if some other condition is satisfied. Perhaps we need to use Lowenheim–Skolem to cut down to a countable model, like Cohen did in his original forcing papers.

Keith Ramsay