

## Re: Countably infinite Hausdorff topology?

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Date: Wed, 15 Sep 2004 06:33:08 -0500

On Tue, 14 Sep 2004 14:04:39 GMT, "shedar" <[nobody@nonesuch.com](mailto:nobody@nonesuch.com)> wrote:

>"Robert Israel" <[israel@math.ubc.ca](mailto:israel@math.ubc.ca)> wrote in message  
>news:ci50j1\$hsf\$1@nntp.itservices.ubc.ca...  
>> In article <[fk11d.6375\\$G03.1882402@news4.srv.hcvlny.cv.net](mailto:fk11d.6375$G03.1882402@news4.srv.hcvlny.cv.net)>,  
>> Stephen J. Herschkorn <[herschko@rutcor.rutgers.edu](mailto:herschko@rutcor.rutgers.edu)> wrote:  
>> >A standard exercise is to show that any infinite sigma-field has  
>> >cardinality at least  $2^\omega$ . That got me thinking about topologies.  
>>  
>> >Does there exist a Hausdorff space whose topology is countably infinite?  
>>  
>> Suppose  $X$  is a Hausdorff space with an infinite topology. In particular,  
>>  $X$  is infinite. Then, with at most one exception, each point of  $X$  has  
>> a neighbourhood whose complement is infinite (i.e. if there were  
>> two points  $x$  and  $y$  whose neighbourhoods all had finite complements, they  
>> could not have disjoint neighbourhoods, violating the Hausdorff  
>> requirement). So let  $x_1$  be a point of  $X$  with an open neighbourhood  $U_1$   
>> whose complement is infinite. Now  $X \setminus U_1$  is also an infinite Hausdorff  
>> space, and the same reasoning applies to it: there is  $x_2$  in  $X \setminus U_1$   
>> with an open neighbourhood  $U_2$  such that  $X \setminus U_1 \setminus U_2$  is infinite.  
>> Moreover, again using the Hausdorff property we may assume that  $x_1$   
>> is not in  $U_2$ . Using induction  
>  
>[See comment (\*) by Shedar below.]  
>  
>> we get sequences of points  $x_j$  of  $X$   
>> and open sets  $U_j$  such that  $x_i$  is in  $U_j$  iff  $i = j$ . Then for any  
>> subset  $S$  of  $\mathbb{N}$ ,  $U_S = \text{union } \{U_j: j \text{ in } S\}$  is an open set which contains  
>>  $x_i$  iff  $i$  is in  $S$ . These constitute a family of distinct open  
>> sets of cardinality  $c$ .  
>>  
>  
>Comment (\*): Correct me if I wrong, but I think "induction" (or technically,  
>recursion on the least limit ordinal)

I don't think "or, technically" is quite right here – there's nothing imprecise or informal about just saying "by induction"; induction

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on the natural numbers is a perfectly respectable thing.

- >would not "cut it". I think one needs
- >to invoke AC (or at least the axiom of countable choice) to get the
- >sequences mentioned above.

Seems to me that you may well be right that some sort of AC is required for the argument as stated. Seems like a slightly silly thing to point out, because people use AC without mentioning it this way all the time.

Seems particularly silly in this case because it's trivial to convert the argument into one that does not use AC (at the expense of converting it into a proof by contradiction): Assume the topology is countable and fix an enumeration of the open sets. Now each time we need to choose an open set with a certain property choose the one with that property that comes first in the enumeration.

("Or technically", use the fact that the first infinite ordinal is well-ordered...)

- >Shedar
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David C. Ullrich