Re: Countably infinite Hausdorff topology?

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From: shedar (*no_one_at_nonesuch.com*)
Date: 09/17/04

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Date: Fri, 17 Sep 2004 12:04:04 GMT
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"David C. Ullrich" <ullrich@math.okstate.edu> wrote in message
news:ir9gk0p7vvkua3budg8ems9n17ch4tqtg6@4ax.com...
> On Tue, 14 Sep 2004 14:04:39 GMT, "shedar" <nobody@nonesuch.com>
> wrote:
>
> >"Robert Israel" <israel@math.ubc.ca> wrote in message
> >news:ci50j1hsf1@nntp.itservices.ubc.ca...
> >> In article <fkl1d.6375$G03.1882402@news4.srv.hcvlny.cv.net>,
> >> Stephen J. Herschkorn <herschko@rutcor.rutgers.edu> wrote:
> >> >A standard exercise is to show that any infinite sigma−field has
> >> >cardinality at least 2^omega. That got me thinking about topologies.
> >>
> >> >Does there exist a Hausdorff space whose topology is countably
infinite?
> >>
> >> Suppose X is a Hausdorff space with an infinite topology. In
particular,
> >> X is infinite. Then, with at most one exception, each point of X has
> >> a neighbourhood whose complement is infinite (i.e. if there were
> >> two points x and y whose neighbourhoods all had finite complements,
they
> >> could not have disjoint neighbourhoods, violating the Hausdorff
> >> requirement). So let x_1 be a point of X with an open neighbourhood
U_1
> >> whose complement is infinite. Now X \ U_1 is also an infinite
Hausdorff
> >> space, and the same reasoning applies to it: there is x_2 in X \ U_1
> >> with an open neighbourhood U_2 such that X \ U_1 \ U_2 is infinite.
> >> Moreover, again using the Hausdorff property we may assume that x_1
> >> is not in U_2. Using induction
> >
> >[See comment (*) by Shedar below.]
> >
> >> we get sequences of points x_j of X
> >> and open sets U_j such that x_i is in U_j iff i = j. Then for any
> >> subset S of N, U_S = union {U_j: j in S} is an open set which contains
> >> x_i iff i is in S. These constitute a family of distinct open

> >> sets of cardinality c.
> >>
> >
> >Comment (*): Correct me if I wrong, but I think "induction" (or
technically,
> >recursion on the least limit ordinal)
>
> I don't think "or, technically" is quite right here − there's nothing
> imprecise or informal about just saying "by induction"; induction
> on the natural numbers is a perfectly respectable thing.
>
> >would not "cut it". I think one needs
> >to invoke AC (or at least the axiom of countable choice) to get the
> >sequences mentioned above.
>
> Seems to me that you may well be right that some sort of AC is
> required for the argument as stated. Seems like a slightly
> silly thing to point out, because people use AC without
> mentioning it this way all the time.
>
> Seems particularly silly in this case because it's trivial
> to convert the argument into one that does not use AC
> (at the expense of converting it into a proof by contradiction):
> Assume the topology is countable and fix an enumeration of
> the open sets. Now each time we need to choose an open
> set with a certain property choose the one with that
> property that comes first in the enumeration.
>
> ("Or technically", use the fact that the first infinite
> ordinal is well−ordered...)
>

1. On the Use of AC:

I agree that by re−casting the previous exposition in some form of Reductio
Ad Absurdum (RAA), one may be able to get away from the need to use AC. It's
just that I feel whenever AC is invoked, it should be explicitly mentioned,
regardless of whether it is being used in its "most intuitive" form (e.g.,
Russell's MP), or in any of its other forms−−it's only fair. After all, it
is not customary to use ZL or WO without directly alerting the reader so. I
ask the same for Russell's MP as well.

<Joke on the "equality of choices">
Motto: All animals are equal but some are more equal than others.
(Orwell)
Thus, we rank the animals as follows:
1. ZL [perfectly fine with explicit mentioning]
2. WO [Wow, it's awesome (secretly, crazy), but it's OK if I mention
it.]
3. <add your own favorite version here and increase the numbering by 1>
...

n. MP,AC (but, of course; no explicit mentioning is needed)
</Joke>

Yet, there is another more subtle point. If one does not use AC, one will be forced to face with the ugly details of "transfinite recursion". See point #2 below.

2. On the Application of the Transfinite Recursion Theorem (specialized to "omega" in this case):

(a) "Definition by induction" or "definition by recursion":

OK, it's just a name. I understand that many mathematicians have been trained (and continue to be trained) to call this mode of defining a function "definition by induction". Indeed, J.L. Kelly calls it so in the "set theory" appendix to his General Topology (1955 Springer, pp.270−271). Paul Halmos also calls this application of recursion "definition by transfinite induction" in his Naive Set Theory (1974 Springer, p.71). While Kelly did not explicitly label his Thm 128 with the moniker "Transfinite Recursion", Halmos labels his version clearly so. Nevertheless, both authors clearly state that "definition by induction" is really an application of the Transfinite Recursion Theorem. (For a precise statement of this Theorem in ZF, see, for example, pp25−27 of Kunen's Set Theory: An Introduction to Independence Proofs, Elsevier 1980).

(b) Transfinite Recursion or AC "to−the−rescue":

It is then clear that when mathematicians define a function "by induction", they are TECHNICALLY appealing to the Recursion Theorem (the proof of which, incidentally, requires transfinite induction). Regardless of whether one call it "by induction" or "by recursion", this begs the following question:

> When defining a function "by indcution", am I appealing to the Recursion Theorem (and hence, it would be more accurate to call it "defining a function by recursion"), or am I simply just waving my hands here because I don't really want to be bothered with the details of the recursion theorem?

In fact, the following 1997 message by Walter Felscher I found on "math−history−list" seems to be saying that part of the history of the various formulation of AC appears to be motivated by the desire to get away from the ugly details of transfinite recursion:

http://mathforum.org/epigone/math−history−list/jeldclehzun/Pine.HPP.3.91.971204161212.10171A−100000@commli

A small paragraph of it is quoted below.
quote

> Up to this day, most mathematicians teaching topics from analysis or algebra (1) talk about sets and their formation (2) mention Zorn's lemma should they need to use it, but (3) assume in perfect naivity that recursively

defined functions exist without requiring a proof of that
fact. This should illustrate the difficulties arising
when it came to understand the even more general
principle of transfinite recursion.

unquote

3. On the equivalence of the Recursion Theorem and some form of "choice":

[I will use the following notation. CC: axiom of countable choice, DC: axiom of dependent choice]

According to p.147 in Discovering Modern Set Theory I by Just and Weese (1996 American Mathematical Society), the Recursion Theorem is not derivable from (ZF + "CC"). In fact, under ZF, the Recursion Theorem is equivalent to DC.

4. Final Thoughts: AC/DC?

Given the available choices for getting to my function, should one appeal to AC (expedient) or to transfinite recursion (with all its "glorious" details and still be bound to DC)? I think I personally will vote for expediency. What do you think?

Shedar