Re: Chances of (random(0,n) + random(0,n) <= m)

Source: http://sci.tech−archive.net/Archive/sci.math/2004−11/0802.html

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Date: 11/03/04

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Date: Wed, 03 Nov 2004 17:21:08 -0500
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In article <76facea6.0411031205.ed74349@posting.google.com>,
sgeos@granicus.if.org (Brendan Sechter) wrote:

> If I am going pick two numbers at random, both from zero to n and add
> them, what is the chance of the sum being less than or equal to m?
>
> This seems like it should be easy, but my math is really rusty.
>
> If n is 5 and m is 3 I can draw this diagram:
>
> | m n
> | 0 1 2 3 4 5
> −−−−−+−−−−−−−−−−−−−−−−−−−−−−−
> |
> 0 | 0 1 2 3 / 4 5
> | /
> 1 | 1 2 3 / 4 5 6
> | /
> 2 | 2 3 / 4 5 6 7
> | /
> m 3 | 3 / 4 5 6 7 8
> | /
> 4 | 4 5 6 7 8 9
> |
> n 5 | 5 6 7 8 9 10

Notice the above table has n + 1 rows and n + 1 columns. So there
are (n + 1)(n + 1) = n^2 + 2 n + 1 entries, each equally probable.

In how many of these is the entry, that is, sum, less than or
equal to m?

Three cases suggest themselves to me.

Case 1: m less than or equal to n.

In this case, the first column has (m + 1) entries with a sum
less than or equal to m. The next column has one less, and so on.

So the number of entries in which the sum does not exceed m is

$$(m + 1) + m + (m - 1) + ... + 1$$

Observe (for k even):

$$1 + 2 + 3 + ... + k = (1 + k) + (2 + (k - 1)) + ... + ((k/2) + (k - (k/2) + 1))$$
$$= (k/2)(1 + k)$$
$$= k (k + 1)/2$$

You can check that that formula works as well for k odd. (There's a story about Gauss to go along with this formula.

So the number of entries meeting the constraint on the sum is

$$(m + 1)(m + 2)/2$$

That is, the probability that the sum is less than or equal to m in this case is:

$$[(m + 1)(m + 2)]/[2 (n + 1)(n + 1)]$$

Case 2: m greater than n and less than or equal to 2 n

The first column with the last entry equal to m is the (m − n + 1)th column.

The entry in the (n +1)th column adding up to m is the entry in the (m − n + 1)th row.

So the number of entries which meet the constraint is

$$(n + 1)(m - n) + (n + 1) + n + ... + (m - n + 1)$$
$$= (n + 1)(m - n) + (n + 1) + n + ... + 1 - [(m - n) + (m - n - 1) + ... + 1]$$
$$= (n + 1)(m - n) + (n + 1)(n + 2)/2 - (m - n)(m - n + 1)/2$$

$$= (n + 1)(m - n)/2 + (n + 1)(n + 2)/2 + (n + 1)(m - n)/2 - (m - n)(m - n + 1)/2$$

$$= (n + 1)(m + 2)/2 + (m - n)(2 n - m)/2$$

So the the probability that the sum is less than or equal to m in this case is:

$$[(n + 1)(m + 2) + (m - n)(2 n - m)]/[2(n + 1)(n + 1)]$$

There's probably a more compact way of writing that.

Case 3: m greater than 2 n

This case is trivial. In all the entries, the sum is less than m. Thus, the probability of the sum being less than or equal to m is unity.

I welcome correction to any of the above, if any is needed.

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Mostly economics:  <http://www.dreamscape.com/rvien/#PublicationsForFun>
r           c
 v         s a           Whether strength of body or of mind, or wisdom, or
  i       m   p          virtue, are found in proportion to the power or wealth
   e     a     e         of a man is a question fit perhaps to be discussed by
    n   e       .        slaves in the hearing of their masters, but highly
     @ r         c m     unbecoming to reasonable and free men in search of
      d           o      the truth.    -- Rousseau
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