Re: Skolem's Paradox and why is math the way it is?

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From: KRamsay (*kramsay_at_aol.com*)
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In article <39d6e584.0411050909.504582a3@posting.google.com>,
troubled6man@yahoo.com (J.E.) writes:
|kramsay@aol.com (KRamsay) wrote in message
|news:<20041104185237.21658.00000013@mb−m14.aol.com>...
|> In article <39d6e584.0410301037.7c9b415@posting.google.com>,
|> troubled6man@yahoo.com (J.E.) writes:
|> |I have taken classes on set theory, my professors lied to me. I own
|> |set theory books, and they use circular logic.
|>
|> You know, I've seen you make this claim of "circularity" a number
|> of times, and I don't remember you ever saying that any specific
|> thing is circular, giving the cycle of dependency you think there is.
|> I think you're mistakenly perceiving a dependency in their presentation
|> that isn't there. A lot of the things you say come off to me as if you
|> are trying to put the cart before the horse, making the meaning of a
|> concept depend on the axioms used to explore it and so on. I think
|> it would make things much clearer if you expressed this kind of
|> complaint specifically.
|
|I wasn't trying to critize all teachers or all books, just the ones
|I've had.

Okay.

| Someone here suggested "Set theory, Logic, and thier
|limitations", but that books assumes set theory (and induction) in the
|metalanguage to define set theory.

I thought that this might be the kind of thing that seemed circular to you. The point is that you need to have an informal understanding of certain concepts before you can develop a formal one. Before you can have a concept of "formula", you need to have a concept of something like "character string", which is a finite sequence of characters. If one intends by "finite" what is usually meant by "finite", this requires having an informal understanding of induction before you even start dealing with formal theories.

Of course, there are people who argue that one should start from
a more naturalistic notion of "formula", and not assume that anything
like induction applies to it.

At the very beginning, then, we have to distinguish between the
informal concept of string and any formal concept of "string" we
might define inside of a formal system. There are several ways to
get the ball rolling, and not everybody is clear about which they
have in mind, partly because it usually doesn't make any difference
in the material to be treated later.

A platonist would ordinarily consider the whole course to be about
certain types of abstract objects that are defined informally (i.e.,
not depending on a formal system). Everybody needs some such informal
concepts (such as "string") to get started, but the platonist accepts
some reasonably rich kind of abstract object as a starting point. The
axioms the platonist states are then meant as truths about those
abstract objects. The whole discussion, viewed this way, relies upon
assuming that the axioms are true in some informally defined domain,
which for some people is hard to believe. But if we do believe, then
we have no further qualms about applying the theorems we prove using
those axioms back to the formal system itself.

A formalist would approach things differently. One possibility would
be to make the whole book formal. The "metalanguage" you mention
would then be formal. The "object language" would be a formal object,
with no assumed connection to the real world. I could prove theorems
in ZFC about ZFC without believing that they have anything to say about
ZFC. Imagine a formalist somehow proves in ZFC that there is a proof
in ZFC of the Riemann hypothesis. A real feather in his cap! But this
doesn't compel him to believe that it's possible to prove the Riemann
hypothesis in ZFC.

Alternatively, a formalist could make some kind of argument in favor
of the formal theory being validly applicable to the original concepts
such as "strings". That would be a departure from dealing with it in
a purely formal way. But formalists need to account for applied math
somehow or other, and however it is that they justify it to themselves
(which necessarily relies upon something other than just the axioms)
might be sufficient to justify applying theorems about the "object
language".

| I think it's just that the
|teachers I had before weren't clear, it seems like something has to
|come first.

It can be difficult to be clear without being philosophically
heavy−handed. This happens when people teach quantum mechanics
in a similar way. One has these several different "interpretations".
One could pick one and teach as if it were so, which can be somewhat
misleading. On the other hand, one could try to teach the various

interpretations along with the more essential material, which could make it confusing. An instructor could try to stick to just the most essential material−− that part you need to understand to be able to compute the probability of observed outcomes.

To be completely "unbiased" and transparent with regard to such possible qualms as doubting that the natural numbers really exist probably seems like too much of a distraction. So the usual approach is basically not to worry about it. The platonist and the formalist will tend to sound the same as they are developing a theory, since deducing consequences from some assumptions typically sounds just like you believe the assumptions to be "actually true" in some domain. And then one can divert discussion of qualms to such venues as sci.math.

| We can assume induction in the langauge and then latter
|show there there is an induction INSIDE the theory as well, so that we
|don't have to use induction outside the theory, but that's very very
|different than proving induction without proving induction. That's
|proving induction in a theory using induction outside the theory.

Yes, *if* you assume induction for your original concept of "string", it's an informal assumption that can't come from inside the theory. Having proven mathematical induction from the axioms, to conclude that it applies to actual strings is a further step, requiring either believing the axioms are correct in some sense or something like that.

| Set
|theory can't do it's own model theory and have every existentionally
|possible set be in the model,

Note that I only have a vague idea of what "existentionally possible set" means. "Existentionally" looks like a cross between "existentially" and "extentionally".

I also don't really know what you think "do it's own model theory" should mean. I don't consider any particular kind of self−application ("do your own model theory", "define your own truth−predicate", and the like) to be an important criterion for a theory. I can see how if one were looking for a theory to be the be−all and end−all of theories, one would need for it to be able to do to itself anything that any other theory could do to it. But the quest for the be−all and end−all of theories seems a bit quixotic. One surely would need to quit focussing on particular formal theories to serve as the be−all and end−all, and look instead at such things as the process by which we develop theories, and try to find the ultimate theory−development process.

For theories with more realistic goals, I would say self−application is a bit like being able to lift all the rocks that one can make. It could be a sign that one is strong. Or it could be a sign that one

has a limited ability to make rocks. IF logic can define its own
truth−predicate, yes. But that's a combination of being strong in some
ways, and being weak in others.

| but that doesn't mean there isn't a
|strong theory that *can* do it's own model theory that has set theory
|(and hence everything based on it) as a component. That's what I'm
|looking for now, and I think the excluded middle is the only thing in
|the way really. There is a subsection of the universe where the
|excluded middle holds, and that's what we call set theory, but it's
|intended models (if it has any) live outside that subsection.

Why?

I mean, I have no big problem with abandoning LEM. Perhaps it's a
bit of a big step to propose that it be abandoned generally... but
constructive mathematics proceeds without making any blanket
assumption of LEM.

It is consistent to assume that there exists a function from some
subset of the integers onto the reals, e.g. the function taking the
indices of Turing machines that compute real numbers to the real
numbers that they compute. (Compute is in the sense of computing
rational approximations to them.) Perhaps something of that nature
would be more agreeable to you. It still doesn't mean that the reals
are countable, however.

|Of course, no one really wants an entirely new system, so I'll look
|for a new system (theory A) without excluded middle from within set
|theory, and then reconstruct the full set theory within that.

Okay, knock yourself out (figuratively speaking).

[...]
|The math classes I took assumed that for every sentance T(x) such that
|for every set X such that for all x (x in X) => ((T(x) is true) or
|(T(x) is false)) there exists a set Y such that for all y (y in Y) <=>
|((x in X) and T(x)), where set and sentance were both undefined", then
|the standard interpretation of set and sentance were "assumed" outside
|the theory, but maybe the set part is fine, but they should have been
|more honest about what was a valid sentance, some professors actually
|wrote "the sky is blue" as an example sentance, with the proviso that
|it is true.

Well, you do realize that a lot of people consider it rather artificial
to go around talking about "what is true inside the theory" and "what
is true outside the theory" as if they were two very different spheres
of discourse? It just sounds like a course being taught as if
a standard realist point of view were valid.

I don't see any problem with sets of natural numbers such as {n : there exists an Oscar award winner who has had n divorces}.

To the extent that the sky being blue or actually being divorced have some degree of grey area, these may be somewhat fuzzily defined. But it's only when you decide to formalize the theory that you need to specify what kind of formulas are allowed.

The problem only comes in when you want to formalize the axioms. If your instructor stated the selection axiom as above without noting that it's an axiom schema, and that the axioms generated by the schema can only have T(x) that is expressible in the language, then they've made a rather technical inaccuracy in the presentation. But if for some specific formula T(x) the above is an axiom in the language of ZFC, it obviously must be a T expressible in the language of ZFC.

Judging by the kinds of questions people ask on sci.math, imposing this kind of technical detail right away tends to be confusing to many students. Moreover, the informal assumption that this axiom continues to be true for any predicate is the more basic assumption than the axiom schema is.

| But this sneaks a truth predicate into set theory that
|isn't supposed to be there, and I know they were smart enough to know
|better, so I'm left to conclude that they did it on purpose.

You're a mighty suspicious guy.

Including the phrase "is true", as you have it above, is often just a kind of verbal flourish. If someone defines "A or B" to mean "either A is true or B is true", that does not mean that they're intending to define "or" in terms of a truth−predicate. Failing to restrict to formulas in the language of ZF is a matter of allowing set theoretical language to intermix with ordinary language.

Don't confuse smartness with being sufficiently persnickety to avoid making technical slips on the order of the things you're describing here, or caring a lot about them. I realize that to you, the fact that the axiom schema in ZFC only guarantees selection for formulas in ZFC seems very important, but that's because of the kind of concern you have for explaining to yourself how models of ZFC can manage to be countable and things of that kind.

NB: the technical error here is ONLY an issue if the instructor in question was stating the selection axiom schema this way *as a part of ZFC*. If they're just stating it as an axiom, then they're just giving an informal axiom.

| And I
|shouldn't have to wait for weeks to get a book that defines formulas
|without assuming set theory first, it's a bit sad that so many people
|do this in a non−rigorous way.

Don't confuse rigor with formality. Only a formalist needs to have
"formula" defined *inside* of a formal theory separately from outside.

|Then there is the whole colloquialness of truth, theorem, theory,
|model, proof, that people use. I don't think they were trying to be
|dishonest there, but it's very very very difficult for students to
|learn when people are using the words different ways.

The responsibility for making an author−reader or teacher−student relationship work is a shared one. I don't think your teachers or the authors of the books you've read have failed you to the degree you're suggesting. Most students, although they may have some difficulties, tend not to get so hung up on the particular issues that you've described. I don't think you need to have regarded it as such an impediment.

|> Mathematicians seem generally, even the ones who are not formalists,
|> to treat the job of deducing consequences from axioms as playing a
|> special role in doing mathematics. It is supposed to be what we can
|> all agree on. I certainly hope that there is no circularity in your
|> set theory books in that part! Your set theory books should contain
|> many theorems that follow definitely from one of the usual sets of
|> axioms for set theory.
|
|If the axioms aren't described clearly enough, it's not much an
|exercise in anything.

Clearly enough for *what*? I don't think you can name any exercise where you are asked to prove a result, and where the reason why it is difficult for you to complete the exercise is that it wasn't clear enough what the axioms were.

[...]
|Lost you on the definitions again. Is a theorem a truth of all models
|or a provable statement of a language (assuming a fixed standard of
|proof)?

If a system of axioms is a formal system, then "theorem" means a well−formed formula that follows from the axioms by the rules of the system. If we simply give a set of statements as axioms, the theorems are the statements that logically follow from the axioms. If the language of a system is understood as being statements about a (variable) model, then this becomes "true in all models", since in that case, for a statement to follow logically from a collection of other statements simply means that it holds for all models in which the premises do.

I had in mind the common situation where one has a first−order theory. In that case, we have the Goedel completeness theorem that says the logical consequences of a set of axioms are the same as the consequences that can be deduced using standard first−order logic.

|If the latter, then what does it MEAN to be "interested in
|whether a theorem follows from the axioms", since they all do?

What I should have written was "whether a statement follows from the axioms", or "whether a statement is a theorem".

In any case, those of us who are not formalists seldom care whether the Riemann hypothesis is a theorem of ZFC or PA or whatever, or whether the twin prime conjecture is a theorem of ZFC. We do care about whether they're true, however. The formalist thinks somehow that these questions are not well enough defined, but everybody else aside from Essenin Volpin as far as I know disagrees.

[...]
|The dependancy was in defining the axioms. What I think you call
|formal (what I'm used to called pure,a s opposed to applied)
|mathematics is about propositional relations, like x is a y, where you
|don't say (or know) what x is or y is or even "is a" is or means, and
|the statement "x is a y" is obviously netihre true or false, it's
|meaningless. But what you do is assume that certain relations BETWEEN
|propositional relations hold, like "for all x, for all y, (x is a y)
|or (y is a x)", then you can consider what other propositional
|relations must ALSO hold that hold INDEPENDANT of any meaning ascribed
|to x, y, or "is a". Then later if a model exists, that means someone
|can make an interpretation where the x's, y's, and "is a"
|propositional relations are interpreted to be mean something, and the
|model is faithful is the axioms (as propositional relations) hold true
|in the model (as meaningful statements), and a theorem of the axiom
|system is a statement in the language of the thoery that is true in
|all faithful models of the axioms. That's how it works for group
|theory,

I don't think so.

I just went over to my bookshelf and opened a group theory textbook at a random page. The theorem there was that the center of the group GL(n,F) consists of the set of diagonal matrices. GL(n,F) consists of the invertible n by n matrices with entries in the field F.

What are the axioms that supposedly define GL(n,F)? We all know what natural numbers are, and what invertible n by n matricies are, but not because there are "axioms" for them. An n by n matrix is a function from {1,2,...,n}x{1,2,...,n} to F; invertibility means that there exists another such one that is its inverse, etc.

The starting point is arithmetic, i.e., knowing what it means to have a natural number n. What complete axiomatization of arithmetic do you have in mind when doing group theory?

| field theory,

I don't really have a book just on field theory as far as I know,
but it occurs to me that one field being algebraic over another
isn't first−order definable. The property of a field extension,
that the overfield is a *finite* dimensional vector space over
the subfield, comes up often. The finiteness intended is what we
(foolishly?) understood as just plain finiteness, not "finiteness
relative to a model of [something]".

| geometry, etc.

How many colors are needed to color the points inside a unit
square, so that no two points of the same color are a distance
of 1/2 apart?

When people doing Euclidean geometry talk about the Euclidean
plane, they are talking about the one that's isometric to R^2,
pairs of real numbers, not an arbitrary model of some first−order
axiomatization of it.

Generally, your statement comes much closer to correct if we
are considering relationships between second−order statements.
But there's no formal deductive system for second−order statements
that captures all the valid deductions that can be made in
second−order logic. Second−order logic also involves referring
to arbitrary subsets of the domain, which is the usual bugaboo
of set theory.

[...]
|> A formalist considers everything above the bottom line to be just a
|> kind of rhetorical flourish. A Platonist will tend to regard the formal
|> side as being just another technique for refining informal reasoning.
|> Not many mathematicians are very much interested in either refining
|> our explanation of what the undefined terms like "set" mean, or justifying
|> the truth of the axioms in those terms, however. Whether a given
|> mathematician believes the axiom of choice tends to be treated as a
|> matter of personal belief.
|
|That's VERY annoying. I took a class in functional analysis where the
|professor actually changed whether the axiom of choice was true
|halfway through the semester, I basically had to go redo everything.

That's a great story, and I agree that that's annoying, at least if
he did it in a way that forced the class to redo work. If he had
meant to do this, he should at least have started with the neutral
theorems (ones not needing choice) and then added the additional ones
that can be proven with choice.

But whether to believe the axiom of choice is "really" true or not has
no necessary connection with whether someone works using it or not. One
formalist who doesn't believe that 10^n exists for each natural number
n has proven results in set theory using all the usual "highfalutin'"

assumptions.

|Hintikka gives a justification about why mathematicians like the axiom
|of choice because it translates the standard interpretation second
|order formulas into equivalent first order formulas, but then he shows
|that that doesn't work in general.

Mathematicians do tend to like instances of quantifier−elimination,
even when they are unaware of the concept of "quantifier−elimination".

| I think that's because he was
|holding onto an "excluded middle" for atomic sentances when he didn't
|need to, but that's his problem, there is no reason I can't assume no
|excluded middle.

Some properties of structures seem simply to be second−order and
not first−order. Whether a graph is connected or not, for example.
I don't see how one can get around it.

Keith Ramsay