Re: Why are reals uncountable yet algorithms countable (long)?

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In article <2vs5a8F2nf0jrU1@uni−berlin.de>, "robert j. kolker"
<nowhere@nowhere.net> writes:
|There is a school of mathematical thought that would require all kosher
|mathematical entitiies to be constructable according to a finite
|algorithm but such a diminished system of mathematics cannot prove some
|really cool useful stuff.

The Markov school in Moscow used to have such a view, but as far as I know they're all gone. I don't know whether there is another such school.

The Markov school was a particular kind of constructivism. Constructivism doesn't require that all mathematical entities be constructible according to an "algorithm", although it turns out that in many contexts an object that can be constructively proven to exist without additional hypotheses is guaranteed to be computable.

It's probably fairly common to think that it's a matter of the "kashrut" of mathematical objects. People seem to think that constructivists decide somehow that they don't like the noncomputable objects that are there. But it's not quite as dopey a point of view as that. The motivation behind it has more to do with trying to get more richly meaningful theorems than with barring exotic mathematical objects.

Constructive mathematics is something like mathematics done without using the law of excluded middle or the axiom of choice. The law of excluded middle is a deductive rule allowing one to conclude "p or not p" for a sentence p. It's equivalent to the rule of double negation elimination, which says that "p is true" is deducible from "p is not false".

It's quite true that there's a lot of useful and interesting mathematics that depends upon the law of excluded middle and the axiom of choice. But there are also a lot of ways around using the law of excluded middle and the axiom of choice. A constructivist might liken ordinary mathematical practice to cooking with some favorite ingredient like beef. There are lots of tasty meals to be cooked with beef, but one doesn't really *need* beef, and some people would say it's generally better to do without it. After getting spaghetti with beef broth, peanut butter and jelly with beef sandwiches, ice cream with beef fat garnish, hot and sour soup with beef curd, and so on, one might feel like trying a few beef−free meals once in awhile to see what it's like.

The axiom of choice was proven by Goedel to be consistent with standard other axioms of set theory, if they are themselves consistent, because there's a subclass of sets where the axiom of choice is valid, as well as the other axioms. In a technical sense, the axiom of choice can't be too completely vital to the applications of mathematics. And there is an embedding of the theory without the axiom of choice into the theory with the axiom of choice and the law of excluded middle both removed. To put it simply, the law of excluded middle is equivalent to double negation elimination, and in a certain sense that's the only job that it is needed for. If one is willing to leave enough double negations in, any statement that can be proven with the law of excluded middle can also be proven without it.

I can't claim to know how well the constructivist project of doing mathematics constructively all or nearly all of the time would work. I can say with confidence that it is at least not as much of a straightjacket as many think it is. In some ways, assuming fewer axioms gives one more freedom, even.

One of the criticisms that seems more plausible to me, in fact, is the idea that constructivism forces one to work in more of a refined way than one wants to. There are lots of different concepts that are nonconstructively equivalent but constructively not necessarily equivalent. Instead of having the one standard classical result, it seems like one often winds up with a family of related results.

Take the intermediate value theorem. As usually stated, it is nonconstructive. Given a function f which is linear on the intervals [0,1/3], [1/3,2/3], [2/3,1], with f(0)=−1, f(1)=1, and f(1/3)=f(2/3)=r, to prove that there is a value of x for which f(x)=0 holds, constructively, we have to prove that either r<=0 (in which case x = 2/3 − r/3(1−r) works) or r>=0 (in which case x=1/3(1+r) does). But r<=0 or r>=0 indirectly uses the law of excluded middle.

One standard substitute is the constructively valid result that if f is continuous on [0,1], f(0)<0<f(1), then for each epsilon>0 there exists an x, 0<=x<=1, such that |f(x)|<epsilon.

But then there are various other results that might be useful in various cases. If f is a continuous function on [0,1] with f(0)<0<f(1), and if on each interval [a,b], with 0<=a<b<=1, there is a value x, a<=x<=b, such that f(x)<>0, then there exists a y such that 0<y<1, f(y)=0.

Keith Ramsay