Re: x^y + y^x <1

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On 27 Dec 2004, Amanda wrote:
>Hello
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>I'd like some hints to prove that, if x,y>0 then x^y + y^x >1. If we
>keep y fixed and define f(x) = x^y + y^x, then, if y>=1, x^y and y^x
>are increasing functions of x, so that f(x) > lim (x−>0) f(x) =1. In
>virtue of the symmetry of the function, we see that the inequality
>holds whenever x>=1 or y>=1.
>But if x and y are in (0,1), I got confused. We have f'(x) =
>y(x^(y−1)) + (y^x)*ln(y). f'(x) −> oo when x−>0+, so f is stricly*
*>increasing on (0,a) for some a<1. we have f'(1) = y + y*ln(y), so f'*
>will have at least one root in (0,1) if y <= 1/e. I think f' has no
>root in (0,1) if y >1/e, but I'm not sure.
>Any hint is welcome.
>Thank you.
>Amanda

Let's restrict to (0, 1) x (0, 1) and suppose WLOG that x > y
(the case x = y is easily disposed of). Then either x >= 1/e
or y < 1/e.

Case 1: x >= 1/e. Then x^y + y^x (for x fixed) increases
as a function of y in the domain 0 < y < x, since

x y^(x−1) + x^y ln(x) >= x y^(x−1) − x^y > 0

using the lemma below. Hence... (consider y −> 0).

Case 2: y < 1/e. Then x^y + y^x (for y fixed) decreases
as a function of x in the domain y < x < 1, since

y x^(y−1) + y^x ln(y) < y x^(y−1) − y^x < 0

using the lemma below. Hence... (consider x −> 1).

Lemma: If 0 < y < x < 1 then x^(1/(x−1)) < y^(1/(y−1)).

Todd Trimble