Re: Embedding theorem and uniqueness?

Source: http://sci.tech−archive.net/Archive/sci.math/2004−12/3565.html

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From: W. Dale Hall (*mailtowd−hall_at_pacbell.net*)
Date: 12/01/04

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Date: Wed, 01 Dec 2004 09:44:58 GMT
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Bill Hobba wrote:
> "W. Dale Hall" <mailtowd−hall@pacbell.net> wrote in message
> news:3x1rd.26844$zx1.25347@newssvr13.news.prodigy.com...
>
>>inquirydog wrote:
>>
>>
>>>Hi−
>>>
>>> According to the embedding theorem, any manifold can be
>>>represented as a surface in a higher dimensional space (ie− the
>>>2−manifold with constant curvature a^2 can be represented as the
>>>sphere of radius a in three dimensional space). My question is− is
>>>the surface unique (up to trivial transformations such as translation,
>>>rotation, and curling up in a yet higher dimensional space such as
>>>when one rolls up a plane to a cylinder). For instance, is there
>>>another surface in three dimensional space that has constant curvature
>>>a^2? Are there other solutions which have constant curvature but
>>>eventually have a singularity at global locations?
>>>
>>> thanks
>>> −I
>>
>>I'm not sure what you mean by "the embedding theorem". There are several
>>embedding theorems for manifolds, from Whitney's theorem embedding n−
>>dimensional manifolds in R^(2n), to Nash's isometric embedding theorem,
>>to a number of refinements.
>
>
> I am aware of the Campbell−Magaard theorem from an interest I have in
> Wesson's STM theory. Would you have references for the other theorems?
>
> Thanks
> Bill
>

The Whitney and Nash results are global results, that is, the manifold
in question is embedded in R^N as a submanifold, without self−

intersections. As far as I have been able to glean from a quick search, having been uninformed about Campbell−Magaard, that result is a local result, meaning that a neighborhood of a point in the manifold is embedded. This is a *far* weaker result than Nash's (no offense intended: one is comparing local and global results: local embedding is simpler than global embedding, no issue about it) : Campbell−Magaard produce a local embedding of an n−dimensional manifold in R^(n+1). If one were to ignore the issue of isometry, this would be a simple application of the implicit function theorem. In addition, it is easy to come up with 2−dimensional examples where such a global theorem cannot hold, even without requiring isometry (every closed n−manifold in R^(n+1) must be orientable, for instance).

For references, do this:

google "whitney embedding"

Whitney's embedding theorem is of considerable importance, for two reasons: the "general position" argument that trivially obtains embeddings in dimensions > 2n, and the "Whitney trick", which enables one to eliminate self−intersections in dimension 2n, as long as n>=3. The absence of a Whitney trick in dimension 4 is of monumental importance in low−dimensional topology. For example, it is responsible for the apparent disparity between what techniques can be applied for high−dimensional (>=5) manifolds, and what works in low dimensions. R(only) H(only) Bing had a Texasism regarding this phenomenon (which I'll paraphrase, not having been present at the creation): "Dimensions >4 are big enough to reason with; Dimension 2 is small enough to spank; Dimensions 3 and 4 are too big to spank, but to little to reason with"

google "nash embedding"

John Nash proved that any Riemannian manifold can be embedded in Euclidean space (with its standard metric) by an isometry. The dimension requirements seem extravagant, but then one needs to satisfy a set of PDEs that are anything but well−behaved; for that, one needs slack (to paraphrase J.R. "Bob" Dobbs).

google "isometric embedding" nash

This last one shows a number of hits that reveal an improved proof of the Nash−Moser isometric theorem; Nash's result requires a dimension of n(3n + 11)/2, whereas a 1987 result of Gunther requires only

max{ n(n+3)/2 + 5, n(n+5)/2 }

while both are quadratic in n (the dimension of the embedded manifold), the former is roughly 3n^2/2 , whereas the latter is roughly n^2/2.

With the exception of the Whitney result, that's more than I really
know. I'm familiar with Whitney's embedding theorem, since I used to
pretend to do differential topology, and have never pretended to be
a differential geometer.

>
>>*You probably meant the sphere's radius to be 1/a: a larger sphere has*
>>*smaller curvature.*
>>
>>*The Gauss−Bonnet Theorem expresses the Euler characteristic of a*
>>*manifold in terms of the integral of the Gaussian curvature. This*
>>*is sufficient to guarantee that the surface has Euler characteristic*
>>*equal to 2, since it must be positive, and all compact connected*
>>*surfaces have Euler characteristic given by* $2 - 2g$*, where g is the*
>>*genus (or number of handles, for a sphere with n handles).*
>>
>>*For higher−dimensional manifolds, a manifold with sectional curvature*
>>*strictly between 1 and 4 is homeomorphic to a sphere.*
>>
>>*Dale*
>>
>
>
>

That's all I know for now.

Dale.