

## Re: Putnam Exam 2004 -- [\*SPOILERS\*]

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Dave Rusin wrote:

- > *The Sixty-Fifth Annual Putnam Exam was held today. I have already*
- > *posted a copy of the questions. Here are as many answers as I have*
- > *at this moment; I have nothing to say about A-5 and B-6.*
- >
- > *If I were the grader I wouldn't give myself full credit for some of*
- > *these answers but I think rather than nail down all the details I*
- > *will wait for someone else to provide the slickest answers, making*
- > *the drudgery unnecessary... (See Problems A-2, A-4, and problems*
- > *with detailed computations such as B-3, B-4, B-5.)*
- >
- > *Overall I thought this was an easier exam than is typical, which*
- > *made the event much more fun for the "average" students taking the test.*
- >
- > *I will add these answers and any other interesting responses to the*
- > *files I make available about Putnam questions; see*
- > *<http://www.math.niu.edu/~rusin/problems-math/>*
- >
- >
- > *From Heron's formula, the area and side lengths of a triangle satisfy*
- >  $16 A^2 = 2 (a^2 b^2 + a^2 c^2 + b^2 c^2) - (a^4 + b^4 + c^4)$
- > *Viewed as a quadratic polynomial in  $X = a^2$ , this function is clearly*
- > *increasing as long as  $X < (b^2 + c^2)$ , which means that the area of a*
- > *triangle increases with the length of any side as long as the angle*
- > *opposite that side is acute ( $a^2 < b^2 + c^2$ ).*
- >
- > *(A student pointed out to me at lunch that this is also obvious from*
- >  $A = (1/2) b h$ , *where the height  $h$  of the triangle varies with  $a$  but*
- >  *$c$  is fixed; the maximum height occurs with sides  $b$  and  $c$*
- > *perpendicular.)*
- >
- > *It then suffices to observe that we can transform triangle  $T_1$  to  $T_2$*
- > *by a sequence of operations which lengthen just one edge at a time. This*
- > *is not entirely obvious since, for example, we are not given that the*
- > *smaller triangle is acute. But it's easy to see in a sketch which I will*
- > *not attempt in ASCII, showing the portion of the positive octant in*

>  $a, b, c$  –space which corresponds to triangles ( $a < b+c$ , etc. ); the acute  
 > triangles are the ones external to three cones around the coordinate  
 > axes ( $a^2 < b^2 + c^2$ , etc.) If one always lengthens the shortest edge  
 > (from among those that need to be lengthened to change  $T_1$  into  $T_2$ ),  
 > then that will always be opposite an acute angle. (An obtuse angle will  
 > always be opposite the largest side, and that cannot be the only side  
 > remaining to be lengthened since  $T_2$  is acute.)

An alternative method: enlarging the second triangle by a constant factor allows us to assume that  $a = a'$ . Draw both triangles on the same base:  $ABC$  and  $A'BC$  say with  $A$  and  $A'$  on the same side of  $BC$ . Draw the line  $L$  through  $A$  parallel to  $BC$  and the line  $L'$  perpendicular to  $L$  at  $A$ . If the second triangle had larger area,  $A'$  would lie beyond  $L$  (opposite side of  $L$  to  $BC$ ). Divide the "beyond" of  $L$  into two quadrants by  $L'$ . If  $A'$  is beyond  $L$  then  $b' > b$  if  $L$  is in one quadrant and  $c' > c$  in the other (here is where we use acuteness).

> A-4.

>

> This is a heavily-used formula when  $n = 2$ :  $xy = (1/4)((x+y)^2 - (x-y)^2)$ . I guess I never really thought about generalizing it before!

>

> It suffices to arrange it so that the right side is a nonzero multiple of  
 > every  $x_i$ ; for then it is a degree- $n$  polynomial and also a multiple of  
 >  $x_1 x_2 \dots x_n$ , so the quotient of the two sides is a constant  
 > (necessarily rational, as can be seen from substituting  $x_1 = \dots = 1$ );  
 > we can then divide through to make the constant be 1.

>

> If we arrange it so that the right side is symmetric in the  $x_i$ , then  
 > it suffices to make sure it vanishes when  $x_1 = 0$ .

>

> For odd  $n$  this is easy: for any subset  $T$  of  $N = \{1, 2, \dots, n\}$ , let  
 >  $S(T)$  = the sum of the  $x_i$  with  $i$  in  $T$ . Then consider

>

>  $X = \sum (-1)^{|T|} (S(T) - S(N-T))^n$ ,

>

> the sum taken over all subsets  $T$  of  $N$ . When we substitute  $x_1=0$  we  
 > obtain an alternating sum of terms of the form  $(S(U) - S(M-U))^n$   
 > where  $M = N - \{1\}$ . Each such term arises exactly twice, once from  
 >  $S(U \cup \{1\}) - S(M-U)$  and once from  $S(U) - S(N-U)$ ; since the  
 > cardinalities

> of  $U$  and  $U \cup \{1\}$  are of opposite parity, these two summands  
 > will cancel. Thus,  $X$  vanishes when  $x_1=0$ .

>

> On the other hand,  $X$  (apparently) doesn't vanish identically; for  
 > example,

> when all  $x_i$  equal 1, we have

>  $c = \sum (-1)^{|T|} (|T| - |N-T|)^n = \sum (-1)^k (2k-n)^n C(n, k)$ ,

> which seems to equal  $(-2)^n n!$  (I computed the first few examples but  
 > didn't prove this formula in general.)

I got

$n! x_1 x_2 \dots x_n = \sum_T (-1)^{n-|T|} (S(T))^n$   
eschewing coefficient  $-1$ . :-)

>

> B-2.

>

> *Another slick one: Let  $S = \{1, 2, \dots, m\}$ ,  $T = \{m+1, \dots, m+n\}$ .*

> *How many functions are there from  $S$  union  $T$  to  $S$  union  $T$ ?  $(m+n)^{(m+n)}$ .*

> *Among those are the functions that send precisely  $m$  elements of the*

> *domain to  $S$ , and the other  $n$  elements to  $T$ . There are  $C(m+n, m)$*

> *ways*

> *to choose which elements go to  $S$ , and once those are chosen there are*

>  *$m^m n^n$  such functions. So  $(m+n)^{(m+n)} \geq (m+n)!/m!n! m^m n^n$ .*

Alternative:

One of the terms in the binomial expansion of  $(m+n)^{m+n}$

is  $\binom{m+n}{m} m^m n^n$  so that  $(m+n)^{m+n} > \binom{m+n}{m} m^m n^n$

--

Robin Chapman, [www.maths.ex.ac.uk/~rjc/rjc.html](http://www.maths.ex.ac.uk/~rjc/rjc.html)

"Lacan, Jacques, 79, 91-92; mistakes his penis for a square root, 88-9"

Francis Wheen, [\\_How Mumbo-Jumbo Conquered the World\\_](#)