Well−orderings of infinite cardinals

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Anyone have a hint on the following problem from Kunen? (This is not homework − it is for my own edification.)

Let R be a well−ordering of an infinite cardinal k. Show (in ZFC) that there exists a subset X of k such that |X| = k and R agrees with the usual ordering (i.e., membership) on X.

Say a set is R−agreeable iff R agrees with the usual ordering on it. I have been able to show the result under the assumption that k is regular (define an R−agreeable set by transfinite recursion), but not in the general case. At first glance, Zorn's lemma does not help: It is possible for there to exist a maximal R−agreeable set whose cardinality is strictly less than k. I have tried starting from the following:

In the case when k is not regular, let m be its cofinality. It is easy to show that there exists nondecreasing cofinal f: m −> k such that f(a) is a regular cardinal for all a in m.

Since we can find R−agreeable sets of cardinality f(a) for all a in m, I was thinking we might be able to cook up some sort of compactness argument. (This was the source of my recent question about subbases and compactness.) However, I haven't been able to get this to work, either.

Suggestions?

For clarity, here are some defintions which I am not sure are standard.

We use the von Neumann definition for ordinals and cardinals.

Given ordinals a and b, the map f: a −> b is cofinal iff the range of f has no strict upper bound in b.

The cofinality of an ordinal a, denoted cf(a), is the least ordinal b such that there exists a cofinal map from b to a.
An ordinal a is regular iff a = cf(a).