Re: Smooth coordinate charts, S^1 help

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I have a similar question that came up in function theory class, which I have tried a long time to figure out (actually, instead of doing my homework, which is not a good thing) − I am stuck trying to understand the relation between angle functions and their complex domains (angle functions I take to be branches of the function arg):

Is there a way to classify certain properties of an open set by properties common to all continuous angle functions on that set?

My approach to this problem −

(A) Arg on a cut plain is continuous / Arg on plane onto (−PI,PI] not continuous. Hence the natural qustion − why is there no continuous angle (by continuous angle I mean continuous angle function) on the whole complex plain? I think the answer is that if there were, then S^1 would be homeomorphic to a closed interval.

(Aa)But a more important natural question is how to prove that all continuous angles on the cut plane are translations of 2PIk of the same form, with all angles counterclockwise between 1,0 and the cut being between 0 and the smallest positive angle of the cut, and similarly clockwise to the negative direction.

(B) An angle function could be anything. I think if you define it as Arg on the cut plain except as Arg+2PI on a non−measurable set, then you might end up with a non−measurable angle function. (I think angle function is like at each point setting an elevator to go to some floor in one infinitely high building, where you might jump to any level an any point). So I was thinking, what does it mean that it is continuous?

(Ba) Locally it means that there are no jumps, and jumps can be only jumps of 2PIk. But that doesn't mean there won't be such jumps altogether, like on different components, or on a strip that is twisted around the origin that winds more than once and does not intersect itself, like a long enough part of a spiral curve, if the curve is thickened into a sufficiently narrow strip.

(Bb) but if that strip varies in width and intersects itself a little, then there is no continuous angle function. how is that proved? It seems that this is a more general case of why there is no continuous angle on the whole plane.

(C) the angle of z can be considered as following− a point t such that z/|z| = (cost,sint). So the angle, which on a open connected component is an open interval, could be thought of as lying on the unit cycle (I don't remember what its called, but I mean the the graph of the function defined on the real line by
t −−> (cost,sint)). In this way the angle as a real variable is homeomorphic to the cycle, or a "revolving circle" so to speak.

(Ca) Choosing any point in the component, the angle at the point is chosen by winding up or down the unit cycle as much as necessary to a point which has the xy plane projection of z/|z|. A continuous angle on a connected open set defined as such at that point could map onto an interval of diameter <= 2PI, like for the cut plane, or even map to the whole real line, for example constructing a strip as described in (Ba) for the image of the exponential function on a radial line.

(D) Following (Ca), I think the reasoning to my question becomes apparent. For example, what kind of open connected sets do not have continuous angles, what kind do, and out of these are there properties common to all of one's continuous angles, like having image w/ diam <= 2PI, and what do these properties mean about the topological properties of the domain? My first guess would be that it has something to do with index, winding number, and homotopy, but I don't know enough about those things yet. But it seems to me that there are some properties being used here, for example in (Ba) that I need but that I haven't defined mathematically. Also, I hope to solve this problem without curves, although I'm afraid the only way to define the necessary properties is by properies of certain curves.

Thanks in advance,
Dani

Tony wrote:
> *I'm having a little trouble with the following problem :*
>
> *Consider S^1 (the unit circle) as a subset of the complex plane.*
Define an
> *angle function on a subset U in S^1 as a continuous function F : U*
−−> R
> *such that e^(i*F(p)) = p for all p in U, where R is the real numbers.*
 Show
> *that there exists an angle function F on an open subset U in S^1 if*
and only
> *if U does not equal S^1.*
>
> *Well I think I have the reverse implication. That is, if U doesn't*

equal
> *S^1, then we can define a branch of the logarithm on U. Then, F(p) =*
Log(p)
> */ i*
> *works. Also, F is analytic so it's smooth (just as a sidenote)*
>
> *But I can't get the forward implication. Suppose there exists an*
angle
> *function F on an open subset U in S^1. Then there exists an F : U*
−−> R
> *continuous such that*
> *e^(i*F(p)) = p for all p in U. I sort of want to say that F is a*
branch of
> *log, and therefore U can't be all of S^1 since we couldn't cut a line*
from 0
> *to infinity, but F isn't a branch of log (it's close). What should I*
do
> *here?*
>
> *Thanks for any help,*
>
> *Tony*