Re: How to find this summation analytically?

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On Thu, 3 Feb 2005, lucy wrote:

> *How to find the following sum analytically?*
>
> *sum(1/(2^k+d), k from 1 to infinity);*
>
>
> *If d=0, we all know that the sum is 1;*
>
> *Matlab gives the following results:*
>
> *>> k=[1:10000];*
> *>> sum(1./(2.^k+0))*
>
> *ans =*
>
> *1*
>
> *>> sum(1./(2.^k+1))*
>
> *ans =*
>
> *0.7645*

[other results snipped]
>
> *Any thoughts?*
>
> *Matlab refuses to compute symbolically the sum:*

simple(symsum(1/(2^k+1), 1, n))
>
> *Can Mathematica and maple do the job?*
>
> *Thanks a lot!*

[Skip the sermon if sermons make you uncomfortable]

"Analytically" has many meanings. You probably meant "in a finite formula, using well−known functions". That of course depends on which functions are accepted as "well−known" − at the low level, one may inquire about elementary functions.
[End of sermon]

As a function of the complex variable d,

f(d) = sum(1/(2^k+d), k from 1 to infinity)

it has obviously simple poles at −2^k, k from 1 to infinity, and is meromorphic in all finite complex plane. Are there elementary functions with these poles?
What comes to mind is Gamma(1−log(−d)/log(2)) in a plane with a suitable cut, but it has extra singularities and "wrong" residues.
(Strictly speaking, Gamma is far from elementary, but it is in many lists of standard special functions.)

Standard or not, the above function f(d) satisfies a functional equation

f(d) = 1/(2+d) + (1/2)*f(d/2) with f(0)=1

By induction, we have an extended functional equation

f(d) = sum(1/(2^k+d), k from 1 to N) + (1/2^N)*f(d/2^N)

for all N=1, 2, ...

Exercise: For abs(c) < 2, f(c) has Maclaurin expansion

f(c) = sum((−1)^n*c^n / (2^(n+1) − 1) , n from 0 to infinity).

In combination with the extended functional equation, we can accelerate the convergence (applying Maclaurin series to c=d/2^N for large enough N), and obtain both lower and upper estimates of the sum if d>0.

Cheers, ZVK(Slavek).