Re: JSH: Heart of dispute, number properties

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From: Arturo Magidin (*magidin_at_math.berkeley.edu*)
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In article <1111599548.838439.15600@o13g2000cwo.googlegroups.com>,
Nora Baron <norabaron@hotmail.com> wrote:

[.snip.]

> So you try to invent another ring which is larger than the algebraic
>integers, but contains a few numbers which are not algebraic
>integers. The only units in this larger ring are 1 and −1.
>
> However it is easy to show that it is not possible to construct such
>a ring for the polynomial
>
> P(x) = 2 x^2 + 5 x + 1.
>
>Here is why. The two roots of P(x) are
>
> r1 = (−5 + sqrt(17))/4 and
>
> r2 = (−5 − sqrt(17))/4.
>
> Now suppose you adjoin one of those roots, say r1, to the ring A
>of algebraic integers: i.e., consider A[r1].
>
*> Note that 1/r1 = r2/(r1*r2) = −r2 / (1/2) = −2 r2. But*
>
> 2 r2 = (−5 − sqrt(17))/2,
>
>which is an algebraic integer.
>
> Therefore r1 has an inverse in A[r1]; it is a unit, but it
>is not +1 or −1.
>
> So if you are going to find a ring with the properties you
>want for this polynomial, it is not going to be a ring which contains
>the algebraic integers.

I don't think this is an accurate restatement. What is being asked is
not that the ring contain no units other than 1 and −1 (this is

already false in all number fields except for most of the imaginary quadratic number fields), but rather that the only −>rationals<− which are units are 1 and −1. This is trivally equivalent to asking that R intersect Q be equal to Z.

Now, there are many, many, many such rings. Bill Dubuque and others have given explicit examples. Pick any algebraic number which is not an algebraic number and such that no power of whom is rational, call it r, and take A[1/r]. That ring has that property.

The real difficulty lies precisely in that there are many, many, such rings, and that while Zorn's Lemma guarantees the existence of maximal such rings, there are many such maximal rings; talking about "the" ring with these properties is nonsense. Your ring A[r1] is probably one such ring. So is the ring A[r2]. But there is no ring which contains BOTH r1 and r2 and satisfies the condition of having intersection with Q equal to Z. Because if a ring contains both r1 and r2, then it contains r1*r2 = (25−17)/16 = 1/2. So A[r1] is contained in some "object ring", and A[r2] is contained in another, but there is no "object ring" that contains both.

IF you further ask, as it was at one point implicitly required, that the ring be −>closed under conjugates<− (if f(x) is irreducible over Q and has at least one root in R, then it has all its roots in R), then we overcome the difficulties: there −>is<− a unique largest subring of the algebraic numbers with that property AND with the property that it intersects Q at Z. But that ring is... the ring of all algebraic integers.

[.snip.]

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"It's not denial. I'm just very selective about
 what I accept as reality."
    --- Calvin ("Calvin and Hobbes")
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Arturo Magidin
magidin@math.berkeley.edu