Re: Coprimeness − I think I'm confused, but I'm not sure

Source: http://sci.tech−archive.net/Archive/sci.math/2005−04/msg00376.html

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- *From*: Matt Gutting <matthewdba@xxxxxxxxxxx>
- *Date*: Fri, 01 Apr 2005 18:46:29 −0500

Arturo Magidin wrote:

> In article <1112387619.8214cb7fca92c72ca9857edb4554270a@teranews>,
> Matt Gutting <matthewdba@xxxxxxxxxxx> wrote:
>
>> In light of some of the comments in the JSH threads
>> involving coprimeness and Z[1/2]:
>>
>> If the definition of coprimeness is something like:
>>
>> To say that 'p and q are coprime in a ring' is to say
>> 'there exist a,b in the ring with ap + bq = 1'
>
> This is true in rings with 1.

Okey dokey. I guess it's difficult for me to work with rings
lacking a unit element because the books I've used (Herstein,
and Rotman), while mentioning that "not all mathematicians
insist that a ring contain a unit element", do in fact insist
for the purposes of the book that rings be required to have
a unit element.

You wouldn't know of a (beginning grad−level) book that discussed
the differences between rings with and without unit element?

then wouldn't 2 and 4 (or any powers of 2) be coprime
in Z[1/2]? Or, more strongly, are any 2 (non−zero) rationals
coprime in Q?

Yes, because they are units and a unit is coprime to anything.

Or am I missing something?

Two elements are coprime if and only if there is no prime ideal
that contains both. By definition, the total ideal is not a prime
ideal.

Ah, I should have been working with that idea − or should that be
'ideal'? :−) You're implying, it seems to me, that I haven't been
thinking in sufficiently abstract terms. (At least, after reading your
discussion, that's what I'm feeling.)

I'll have to go reread Rotman and see what he says about ideals.

Thanks!

Matt

In rings with 1, maximal ideals are prime; and (assuming the axiom of
choice, at least) every proper ideal is contained in a maximal
ideal. So for rings with 1, "there is no prime ideal containing both
elements" is equivalent to "there is no maximal ideal containing both
elements".

If u is a unit, then any ideal that contains u must contain all of R;
so there is no maximal ideal that contains u; so there is certainly no

maximal ideal that contains u and <anything else>.

Since 2^n is a unit in Z[1/2], 2^n is coprime to everything in Z[1/2]. Since a nonzero rational is a unit in Q, a nonzero rational is coprime to everything in Q. In fact, the only pair of rationals which are not coprime are 0 and itself.

To get from "no maximal ideal contains both" to the expression as a linear combination, just note that the ideal (p,q) is certainly an ideal that contains both p and q. Since every proper ideal is contained in a maximal ideal, and no maximal ideal contains both p and q, it follows that (p,q) = R. That means that every element of R (in particular 1) lies in (p,q). The elements of (p,q) are exactly all elements of the form   px + qy for some x and y in R. So there exist a and b in R such that 1 = ap + bq. So in fact p and q are coprime if and only if (p,q)=1.

In −>some<− rings, being coprime is equivalent to "having no common divisors"; e.g., in Z, and in the ring of all algebraic integers. In some rings, it is not. For example, in Z[x], x and 2 have no common divisors, but (x,2) is not the total ideal. Likewise, in Z[sqrt(−5)], 2 and 1+sqrt(−5) have no common divisors, but are not coprime.

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