Re: Problems I have with 1.999...=2

Source: http://sci.tech−archive.net/Archive/sci.math/2005−05/msg01723.html

---

- *From*: abe.buckingham@xxxxxxxxx
- *Date*: 10 May 2005 01:49:42 −0700

This is an old debate, and frankly it look me several years before I firmly came to grips with it myself. My understanding came in three parts, firstly what our motivation was for these infinite decimal expansions, secondly what exactly we ment when we said that 'it goes on forever' and finally dealing with the equality itself.

As best I understand it we choose to define 0.999.. = 1 (or 1.999.. = 2) because we're trying to model the real number line from classical geometry. I assume you're familiar with the irrationality of Pi, or the square root of two, which both come from simple geometric objects from compass and ruler geometry. Now we know that in science we have certain limitations on how accurate we can take measurements and in almost all practical situations this means estimating using a rational number. We never take our our ruler, put it against the side of a building and then go 'yep, that's exactly Pi meters long' but at the same time, we realize that when we're modeling such problems it's convienent to assume that Pi is an easily constructed numbers and we want whatever system we're working in to be able to describe that number. After all, it's not hard to imagine a cicle of radius 1, and we'll want a way to describe it's area.

Now, considering that context and what we're trying to describe with our numbers, we come to the problem of what these infinite decimal expansions mean, and how exactly they relate back to our geometric view. Now I assume you're comfortable with the rational numbers since we seem to use them every day to describe pretty much everything, and in fact from a scientists point of view, the rationals are exactly the kind of data we'd expect to be looking at when we actually take measurements. Also, we know that rationals are dense on the real number line, no matter what two points on the line we pick we can find a rational snuggled up close, in fact, arbitrarily close. We could, potentially just pick out a list of rationals that always gets closer to the quanity we're attempting to describe until it's of sufficient accuracy for our purposes.

Here comes the leap − what if we don't know in advance how accurate we're going to require our estimates? What if the degree of accuracy is arbitrary? Well in this case, our list is going to go on forever, since we already know that no rational number exists to describe such a simple quantity like the square root of 2. So now, to describe these special spots on the real number line, we are forced to use infinite sequences of rational numbers.

I assume you're also familiar with how decimal numbers are representations of infinite sums. An infinite sum is normally defined to be an infinite sequence of numbers with each step of the sum taking up a place on the list. So 0.9999... would actually be a shorthand for the sequence (0, 0.9, 0.99, 0.999, 0.9999, ...). Notice that these infinite lists only contain rational numbers for our purposes here so naturally we don't see any objection to the entries and we understand that they must be infinitely long because we don't know how much precision we're going to require. Naturally we only consider lists of this kind that estimate a point on the real number line not some arbitrary list. As far as I understand these special sequences are called convergent sequences and they have the special property that as we go along the distance between members on the list constantly diminishes after a certain point. More formally given a sequence S whose n−th position is called s_n, that for any any arbitrary amount of error, say e, that their exists a k such that for all m > k that the absolute value of s_m − s_m+1 is less then e. That is to say the sequence is always accurate enough for our purposes no matter how accurate we require them to be. Notice we aren't doing anything fuzzy here by disallowing certain sequences, after all we disallow certain types of fractions like 5/0 just because they screw things up in the same way we disallow sequences that mess things up − after all we're trying to model the number line.

Ok so now comes the funky part, these infinite sequences can cause us some trouble. In fact, many of them describe exactly the same point on the real line. Consider a simple example like (0, 0.9, 0.99, 0.999, ...) and (0.9, 0.99, 0.999, ...) which seem to be approximating the exact same point on the line. After all, the only difference is the first term. We could also do something like take either of these sequences and skip every second term or every third and still get a list that estimates the point on the line we're looking for exactly as the original list did. Which list should we choose?

Well another thing about rational numbers is that sometimes fractions run into the same problem. I mean when we say 1/2 we mean the same thing as when we say 2/4, or 4/8 right? Even though they have different members, and aren't *exactly* identical in how we represent them, we do realize that for practical purposes they are the *same*. Why do we say they're the same? Well because those other ways of writing 1/2 have the same properties as the other ways of writing it (IE we'd add, subtract, multiply and divide them as usual) and we have a simple rule to tell when two representations depecit the same number. For rational numbers we say two representations, we'll call them a/b and c/d, are the same if and only if a*d = b*c. Take a minute and convince yourself of this if you're uncertain.

So notice, even though the fractions we use to describe the rationals sometimes overlap we didn't throw them out, we just made a simple rule to tell when two are the same. The real numbers and our infinitely long lists are no different. Our problem is to make sure that we have a simple rule telling us when two of these sequences represent the same point on the line. Mathamaticians do this by defining what's called an 'Equivilence Relation' on the set of convergent sequences. An equivilence relation is an abstraction of what we mean when we say two things are the same. For example, you might only be interested in how long something is, or what colour it is, or what species. Each of these ideas captures to some degree the idea of 'sameness' depending on the context, but would still fail to be *exactly* the same in every concievable way. But as we can see with the rational numbers we have a useful motivation for letting some objects that aren't exactly the same be considered the same in our context.

Equivilence relations have some rules, namely x = x, x = y implies y = x and finally if x = y and y = z then x = z. Lets try to build your intuition by replacing the = sign with the phrase "is the same colour as". Naturally if x is red, then x is the same colour as x. Moreso, if x is the same colour as y, then y is the same colour as x. On top of that, if x is the same colour as y, and y is the same colour as z, then x must naturally be the same colour as z. It's a simple example, but we can see how this captures the idea of sameness and moreso, these are the three rules we use in algebra to make substitutions, prove equality, and otherwise exploit all the advantages that sameness gives us.

These three simple rules also do something else very clever. We could say that Partition a group up into catagories of similar things (based on context) and say anything in the same little group is 'the same' as everything else in the group. So we could organize our wardrobe by colour for example. Now considering these groups and our three rules, we can see clearly that if x is in a group, then x is in that group, and if x is in the same group as y, then y is in the same group as x. Morso, if x is in the same group as y, and y is in the same group as z, then of course x is in the same group as z.

Lets apply this idea to the rational numbers. For the rationals we had fractions, and sometimes they would be the same number like 1/2 and 2/4. We notice these are just pairs of numbers, split by a / sign with the special property that in the integer number system 1*4 = 2*2. So we take all of the numbers written in the form p/q with q not 0 and given any two of these objects we group a/b and c/d together if and only if a*d = b*c. So we've taken all of the objects of the form p/q and grouped them into bunches based on this rule.

Now for the real numbers we have a whole bunch of convergent sequences some of which get arbitrarily close to the same point on the number line. What rule should we use to bunch them up? We would say that given some arbitrary error, say e, which for convinence we always say is greater then 0, that the distance between the terms of our two sequences eventually is lower then e. So we have two convergent sequences Q (with terms of the form q_n) and R (with terms of the form r_n), we would say that Q = R if and only if their exists a positive interger k such that for all m > k the absolute value of Q_m − r_m < e. Remember that e is arbitrary so no matter how small the error is, we'll always be able to find terms of Q and R that satisfy these properties. We now partition the set of convergent sequences into groups based on this rule, and say that everything inside these groups is the 'same' in our context, which is the real number line we've been trying to model all along. Since this forms a nice partition, they follow the three rules of an equivilence relation, and we can suddenly do all the algebra we're familiar with with a few more simple explainations of what we mean by 'adding' and 'multiplying' these sequences together. I'll let you discover those on your own.

So finally we can see how we've come full circle, we've defined what we mean by these infinite decimal expansions and from the context of geometry slowly built up the properties we wanted them to have using the rational numbers and our own reasoning. Since the above defines precisely what we mean by two of these infinite sequences being the same, we now turn to the question of 1.999... =2. In this case we have two sequences, the first being (1, 1.9, 1.99, 1.999, 1.9999, ...) and the other being (2, 2, 2, 2, 2, 2, 2, 2, ...). Both of these are convergent sequences, so we know they're allowable, the only question left is do they both represent the same point on the real number line? So we turn to our equivilence relation that says if we subract these two term by term, given some arbitrarily small number e, can we find a spot where this difference is less then e.

Lets consider what happens when we subtract these two sequences term by term. We get a new list, that goes like this (1, 0.1, 0.01, 0.001, 0.0001, ...) Now no matter what e I chose to start with, I could always find a point on that new list that is less then e since e is greater then 0 and fixed. This means we would bundle (2, 2, 2, 2,...) and (1, 1.9, 1.99,...) into the same group and would indeed call them equal in the context we're discussing. I hope this answers you're question.

.

- ***Follow−Ups***:
    - ***Re: Problems I have with 1.999...=2***
        - *From:* Kirby Cook
    - ***Re: Problems I have with 1.999...=2***
        - *From:* Shmuel (Seymour J.) Metz
    - ***Re: Problems I have with 1.999...=2***
        - *From:* Randy Poe
- ***References***:

- ***Problems I have with 1.999...=2***
    - *From:* Kirby Cook

- Prev by Date: ***Re: measurable set***
- Next by Date: ***Re: Weak and strong convergence in D'(X)***
- Previous by thread: ***Re: Problems I have with 1.999...=2***
- Next by thread: ***Re: Problems I have with 1.999...=2***
- Index(es):
    - ***Date***
    - ***Thread***