Trisecting and angle after infinite steps

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- *From*: tuanglen@xxxxxxxxxxx
- *Date*: 18 May 2005 18:12:02 −0700

I'm trying to solve an infinite series problem, but I'm stuck. It's not a deep mathematical mystery. It's just a problem in a calculus book I'm reviewing. It has been more than 20 years since my last "homework" problem, but I try to review this stuff from time to time on my own so I don't lose it as most of my classmates from those days have long since done.

[This is from p. 651 in Howard Anton's new 8th ed. Calculus, which is generally a very good book for self study]:

"Suppose an angle theta [the one in the diagram has its 'initial edge' at 3 o'clock if you imagine a clock face and its other at roughly 11 o'clock] is bisected to produce ray R1 [shown going from the origin off in the roughly 1 o'clock direction]. Then the angle between R1 and the initial side is bisected to produce ray R2 [shown as a ray going off in the roughly 2 o'clock direction, or theta/4.] Thereafter, rays R3, R4, R5... are constructed in succession by bisecting the angle between the preceding two rays [R3 is drawn between R1 and R2 at about 1:30, R4 between R2 and R3, etc. converging on a limit ray at theta/3]. Show that the sequence of angles that these rays make with the initial side has a limit of theta/3."

Okay, if I'm calculating correctly the angle of these rays, expressed as a fraction of theta, creates the following sequence:

1/2, 1/4, 3/8, 5/16, 11/32, 21/64, 43/128, 85/256, 171/512, ...

This can be expressed as a recurrance relation where each term is the average of the previous two (using parentheses to denote subscripts):
X(n) = [X(n−2) + X(n−1)]/2
initialized by the first two terms 1/2 & 1/4.

Since X(n) converges as n−>inf, I figured that X(n), X(n−1), and X(n−2) would all approach equality (eventually), so I tried substituting the limit into the above equation:

L = [L + L]/2

Doh!

If I could create a closed form expression for term X(n), I could take the limit as n−>infinity. If you call the initial 1/2 the n=1 term, the denominator is clearly 2^n. Then I noticed that the numerators had the pattern

Numerator(n) = 2*Numerator(n−2)+Numerator(n−1)

Combining that with the 2^n denominator and simplifying led to:

X(n) = [X(n−2) + X(n−1)]/2

Doh! That approach just led me back to the "average of the previous two terms" statement.

Well, if X(n) converges on 1/3 (i.e. theta/3), and the denominator is 2^n, the numerator would have to converge on (2^n)/3. Proving that it did would solve the problem, so I tried just dealing with the numerator.:

1, 2, 3, 4, 05, 06, 07, 08, 009, ... <−− n
1, 1, 3, 5, 11, 21, 43, 85, 171, ... <−− numerator of X(n)

>>From here on, I'll call the numerator of X(n) "A(n)", so
X(n) = A(n)/2^n.

I assume that the way to prove that A(n) approaches (2^n)/3 as n approaches infinity is to write a closed form expression of A(n) and take its limit as n−>inf, but I can't see how to write the closed form. For example, for n=1..5 I get:

A(n) = 2*A(n−2) + A(n−1)
1 = 1
1 = 1
3 = 2*1 + 1
5 = 2*1 + 2*1+1
11= 2*(2*1+1) + (2*1 + 2*1+1)

With the recursively tangled additions and multiplications, I don't see how to collapse it into a simple closed form for term A(n).

I found other relationships, too.

If n is even
A(n) = A(n−1) + A(n−2) + A(n−3) + ... A(1)

If n is odd
A(n) = A(n−1) + A(n−2) + A(n−3) + ... A(1) + 1

Unfortunately, I don't see how to collapse this into a simple closed form for A(n).

I'd be grateful for any suggestions at this point, and not just for
this specific problem, but for any general suggestions that might occur
to you as you read the above. I'm trying to hone my skills, not solve
any particular problem.

Thanks.

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- ***Follow−Ups***:
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    - *From:* Michael Jørgensen
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    - *From:* The Last Danish Pastry

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