Re: Rotated square in space

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- *From*: rusin@xxxxxxxxxxxxxxxxxxxxxx (Dave Rusin)
- *Date*: 9 Jun 2005 21:06:57 GMT

I'm coming in late here but I see there was a question about
computing coordinates of points of a square in R^3 knowing their
images under projection to a plane from a single (unknown) light source:

In article <3glfh2Fd4s8dU1@xxxxxxxxxxxxxxx>,
Claudio Grondi <claudio.grondi@xxxxxxxxxx> wrote:

>So your question was about transformation of
>points of a square on a quadrangle which
>results from 3D perspective transformation,
>right?
>
>This is another kind of question as your original
>one:
>> I have this projection:
>> p1 0,5 p2 5,0 p3 15,5 p4 10,15
>> and now I want p1,p2,p3,p4 in 3d world coordinates.

I took the perspective (ahem) that the light source O = (x,y,z)
and the points P_1 = (0,5,0), etc., can be connected by lines
which contain the actual vertices Q_i = P_i + t_i (O − P_i) which
are located some fraction t_i of the way between O and P_i .
Any choice of O and the t_i will give the right projection, but
the OP wants the Q_i to lie on a square. This requires
(1) consecutive vectors v_i = Q_{i+1} − Q_i must be perpendicular
(2) the normals v1 x v2 and v3 x v4 to the triangles Q1 Q2 Q3 and
Q3 Q4 Q1 must be equal
(3) consecutive sides must be of equal length: v1^2 = v2^2

Conversely if (1) is met then Q1 Q2 Q3 and Q3 Q4 Q1 are right triangles,
and if (2) is met then their normals are in particular parallel, meaning
the right triangles are parallel, meaning (since they share vertices) they
are coplanar. So then using (1) again we know the Q_i form a planar
rectangle; (3) makes it square.

So it is easy to generate a set of polynomial equations which precisely
describes the constraints on the unknowns x,y,z, t1,t2,t3,t4. Here it
is done in Maple:

```
p1:=[0,5]: p2:=[5,0]: p3:=[15,5]: p4:=[10,15]:
for i to 4 do (P||i) := [op(p||i), 0]: od:
for i to 4 do (Q||i) := P||i + (t||i)*([x,y,z]−P||i):od:
for i to 4 do v||i := expand( Q||(i mod 4 + 1) − Q||i ): od:
for i to 4 do va:=v||i:vb:=v||(i mod 4 + 1): ip||i:=add(va[j]*vb[j],j=1..3):od:
for i to 3 do ap||i :=
v1[i mod 3 +1]*v2[i−2 mod 3 + 1] − v3[i mod 3 +1]*v4[i−2 mod 3 + 1] : od:
ap0:=add(v1[i]^2,i=1..3)−add(v2[i]^2,i=1..3);
```

Actually there is a nondegeneracy condition we should add: the equations admit the solution t1=t2=t3=t4=1 , i.e. the "square" is at O . We can use the algebraic geometers' trick to make the t_i unequal to 1 by adding additional constraints involving additional variables u_i : insist that (1 − t_i) u_i = 1 and then t_i cannot be zero. Likewise we don't want the "solution" with z = 0. Again in Maple:

```
for i to 4 do bp||i:=(1−t||i)*u||i − 1 : od: bp0:= z*w − 1 :
```

For the given points, the full set of constraints is then

```
(1−t1)*u1−1, (1−t2)*u2−1, (1−t3)*u3−1, (1−t4)*u4−1, z*w−1,
(−x*t1−5*t2+x*t2+5)*(5*t2−x*t2−15*t3+x*t3+10)+(5*t1−y*t1+y*t2−5)*(−y*t2−5*t3+y
*t3+5)+(−z*t1+z*t2)*(−z*t2+z*t3), (5*t2−x*t2−15*t3+x*t3+10)*(15*t3−x*t3−10*t4+
x*t4−5)+(−y*t2−5*t3+y*t3+5)*(5*t3−y*t3−15*t4+y*t4+10)+(−z*t2+z*t3)*(−z*t3+z*t4
), (15*t3−x*t3−10*t4+x*t4−5)*(10*t4−x*t4+x*t1−10)+(5*t3−y*t3−15*t4+y*t4+10)*(
15*t4−y*t4−5*t1+y*t1−10)+(−z*t3+z*t4)*(−z*t4+z*t1), (10*t4−x*t4+x*t1−10)*(−x*
t1−5*t2+x*t2+5)+(15*t4−y*t4−5*t1+y*t1−10)*(5*t1−y*t1+y*t2−5)+(−z*t4+z*t1)*(−z*
t1+z*t2), (5*t1−y*t1+y*t2−5)*(−z*t2+z*t3)−(5*t3−y*t3−15*t4+y*t4+10)*(−z*t4+z*
t1), (−z*t1+z*t2)*(5*t2−x*t2−15*t3+x*t3+10)−(−z*t3+z*t4)*(10*t4−x*t4+x*t1−10),
(−x*t1−5*t2+x*t2+5)*(−y*t2−5*t3+y*t3+5)−(15*t3−x*t3−10*t4+x*t4−5)*(15*t4−y*t4−\
5*t1+y*t1−10), (−x*t1−5*t2+x*t2+5)^2+(5*t1−y*t1+y*t2−5)^2+(−z*t1+z*t2)^2−(5*t2
−x*t2−15*t3+x*t3+10)^2−(−y*t2−5*t3+y*t3+5)^2−(−z*t2+z*t3)^2
```

These polynomials generate an ideal I in the polynomial ring Q[x,y,z,w,t1,t2,t3,t4,u1,u2,u3,u4] ; we would like the simplest possible description of the intersection of this ideal with the subring Q[x,y,z,t1,t2,t3,t4] (since we don't really care about w or the u_i, except for their existence!). I had Magma do this for me:

```
Q<t1,t2,t3,t4,u1,u2,u3,u4,w,x,y,z>:=PolynomialRing(RationalField(),12);
I:=ideal<Q| ... generators above ... >;
J:=EliminationIdeal(I,{t1,t2,t3,t4,x,y,z});
Groebner(J);
```

The answer is that the ideal is equally well generated by just a few simple polynomials:

```
t1 − 5/3*t4 + 2/3,
t2 − 2*t4 + 1,
t3 − 4/3*t4 + 1/3,
x − 1/3*y + 5/3,
y^2 + 26*y + 9/10*z^2 − 155
```

So these are the constraints on our variables. Obviously we can choose t4 and y as we wish (as long as y is between −31 and 5) and compute x, z, t1, t2, t3 from them. (The fact that z appears only to an even exponent is to be expected because of the mirror symmetry with respect to the plane z=0 .) With, say, t4 = 2/3 and y = 2 we get a square with side−length (10/3) sqrt(2) whose vertices are at

(−4/9, 11/3, 4/9*110^(1/2)), (3, 2/3, 1/3*110^(1/2)),
(55/9, 10/3, 5/9*110^(1/2)), (8/3, 19/3, 2/3*110^(1/2)) ;

here the light source is at (−1, 2, 110^(1/2)).

Having the full algebraic solution in hand I see there is probably some way to use in advance some of the geometric degrees of freedom to reduce the size of the set of constraints. One could also pre−process the data to put two vertices at (say) (0,0) and (1,0); perhaps it would be possible to work this all out symbolically, then, giving t1, t2, t3, x, and z^2 in terms of t4, y, and the four coordinates of the other two points in the quadrilateral projection.

dave
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- ***References***:
    - ***Rotated square in space***
        - ◊ *From:* Luiz Borges
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        - ◊ *From:* Luiz Borges
    - ***Re: Rotated square in space***
        - ◊ *From:* Claudio Grondi

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