Re: Compact subsets of {0,1}^N

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- *Date*: Fri, 1 Jul 2005 00:55:50 −0700

From: David C. Ullrich <ullrich@xxxxxxxxxxxxxxxx>

Summary of proof and comments presented by D.C. Ullrich regards

closed without isolated points, nonnul K subset S = {0,1}^N
==> K homeomorphic S

> If a is a _finite_ sequence of 0's and 1's let S_a be the set of
> all elements of S that begin with a. Note that we're including
> the case where a is the empty sequence, ie a sequence of length 0:
> if a is the empty sequence then S_a = S.

> Let B be the tree of all finite sequences of 0's and 1's, and let
> T be the tree of all finite sequences a such that S_a intersect K
> is nonempty.

> Define a map F from B onto T by induction:

> If a is the empty sequence let F(a) = a.
> For each a in B, of length n, let a_0 and a_1 be the two extensions
> of a of length n + 1 (a_j consists of a followed by a j.)
> Now if a is in B and F(a) has already been defined, define
> F(a_0) and F(a_1) as follows: Let b = F(a).

Since K has no isolated points and S_b is open, K intersect S_b must
contain at least two points. So b has at least two extensions in T.
Hence there exists an extension b' of b, with length(b') = m, say,
such that
(i) b' is the only extension of b in T of length m and
(ii) b'_0 and b'_1 are both elements of T.
Define F(a_0) = b'_0 and F(a_1) = b'_1.

(Less formally: Since K intersect S_b contains more than one
point, the node b in T must _eventually_ split in two, at
some level below the level of b. Map a_0 and a_1 to the two
nodes in T at the first level where b splits.)

> It follows that if a' is an extension of a then F(a') is an
> extension of F(a), so F induces a map f:S −> S in a natural

> way (given a in S, f(a) is the unique element of S such that
> f(every finite initial segment of a) is an initial segment of f(a).

> It's easy to see that f is continuous (given a and n there exists
> m such that if a' agrees with a in the first m places then f(a')
> agrees with f(a) in the first n places).

f is 1−1: Say b, c are in S and b <> c. There is a first place at which b and c differe, which says that there is a (possibly zerp−length) finite string d such that d_0 is an initial segment of b and d_1 is an initial segment of c, or vice versa. Now by the construction of F, there exists e such that F(d_0) = e_0 and F(d_1) = e_1, so F(d_0) <> F(d_1). But F(d_0) is an initial segment of f(b) and F(d_1) is an initial segment of f(c), so f(b) <> f(c).

f maps S into K: Say a is in S. Let's say a[:n] is the finite string consisting of the first n elements of a. Now by construction for every n there exists b^n in T such that F(a[:n]) = b^n, note that the length of b^n is at least n. Since b^n is in T, for every n there exists k^n in K such that b^n is an initial segment of K.

Now by construction f(a) agrees with b^n in the first n places, and b^n also agrees with k^n in the first n places, so f(a) agrees with k^n in the first n places. This says that k^n converges to f(a), and hence f(a) is in K since K is closed.

f maps S onto K: Let's say that a finite string t in T is an n−sibling if t has length n and there exists t' in T such that t' <> t, t' has length n, and t and t' agree in the first n−1 places. Say that t in T is a sibling if it is the empty string or an n−sibling for some n. By construction the range of F is exactly the set of all siblings.

Now suppose that k is in K. Since K has no isolated points there exists a cofinal sequence of initial segments of k, each of which is a sibling. Since each sibling is in the range of F it follows that k is in the range of f.

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The fact that f maps S onto K is exactly the part that I originally said I hoped was obvious, because a formal explanation might be a little long. Ok, if it's obvious to me then I should be able to write down a formal explanation − here we go (I'll give an illustration of why every element of K starts with F(0) or F(1) after the proof):

Borrowing notation from Python, if n <= length(t) let's say that t[:n] is the initial segment of t of length n. Now say t' is a child of t if n = length(t) = length(t') − 1 and t'[:n] = n. This "child of" relation makes T into a tree in which each node has one or two children.

Say t' is a descendant of t if t' is a child of t or a child of a child of t or etc. Say t' starts with t if n = length(t) <= length(t') and t'[:n] = t. So of course t' starts with t if and only if t' is a descendant of t; in the one case we're thinking about actual sequences and in the other we're thinking about the tree structure.

Ok. The first fact is that every element of T has more than one descendant. This is where the fact that K has no isolated points is used: Say t is an element of T, choose k in K so that k starts with t (the existence of such a t is exactly the definition of T), and consider a sequence of distinct points of K that converge to k. By the definition of convergence, all but finitely many of those points of K start with t; in particular there exist at least two elements of K that start with t, which says that t has at least two descendants.

So every element of T has at least two descendants, but of course an element of T can have only one child. Say t is in T. Define split(t) = t', where t' is the element of T of minimal length such that t' is a descendant of t and t' has two children.

As before, let a_0 and a_1 be the two sequences consiting of a followed by a 0 or 1, respectively. The definition of F was this: F(empty sequence) = empty sequence, and F(a_0) = split(F(a))_0, F(a_1) = split(F(a))_1.

It follows that the range of F is exactly the empty sequence plus all siblings in T: one direction is immediate, and for the other direction, given t in T which is a sibling, trace the chain of ancestors up the tree until you come to an ancestor of shorter length which is also a sibling or the empty sequence. (If t^ is a proper ancestor of t of maximal length such that t^ is also a sibling then t^ is in the range of F by induction on the length and then the definition of F shows that t is in the range of F. On the other hand, if no proper ancestor of t is a sibling then t = split(empty sequence) and it also follows that t is in the range of F.)

Now fix k in K. The second key fact is this:
infinitely many initial segments of k are siblings.
(We need to show that for every N there exists
n >= N such that k[:n] is a sibling. Fix n. Since
k is not an isolated point of K there exists k'
in K with k' <> k such that k'[:N} = k[:N}.
Since k' <> k there exists n > N such that
k'[n] <> k[n] (where s[n] denotes the n−th term
of s. Choose n minimal with this property; then
k[:n] is a sibling, because k[:n] and k'[:n]
are unequal but they are both children of
k[:n−1].)

So every element of K has infinitely many initial
segments which are siblings. Hence every k in K
has infinitely many initial segments which are in
the range of F, and hence k is in the range of f.

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Why every element of K starts with F(0) or F(1):
It could happen, say, that every element of K
starts with 01101, but not every element of K
starts with 011010 and not every element of K
starts with 011011. If that happens then
every element of K starts with 011010 _or_
011011, and the definitions show that 01101
= split(empty sequence), hence F(0) = 011010
and F(1) = 011011.

Another way of looking at the whole thing − much less formal,
but it may make you realize suddenly why it's all obvious:
Draw a picture of the tree T: An infinite tree where every
node has one or two children and every node has more than
one descendant. Any time you see a chain of nodes each
with only one child, imagine collapsing that chain into
a single node in a new tree. Then that new tree is a
complete infinite binary tree − any two complete infinite
binary trees are obviously isomorphic, and the F above
(translated to our new tree) is just the obvious isomorphism.

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