Visualizing a measure zero cover of the rationals

Source: http://sci.tech−archive.net/Archive/sci.math/2005−08/msg05476.html

---

- *From*: Jim Burns <burns.87@xxxxxxx>
- *Date*: Sun, 28 Aug 2005 13:07:10 −0400

My impression is that covering all the rationals with
intervals with a total length of epsilon > 0 is not a very
controversial result, probably a very basic one in measure
theory. The rationals are countable, so we can list them.
Cover the first rational with e/2, the second with e/4, ...

My problem arose when I tried to visualize this epsilon−sized
covering, spread out everywhere and nowhere. Much later,
it occurred to me that the scheme I was using wasn't really
intended for easy visualization; rather, it was for easy theorem
proving.

I've come up with another scheme that I like. It uses the
continued fraction expansions of the rationals in (0, 1),
assigning half the length to the points on the top level
{ 1/2, 1/3, 1/4, ...} and the other half of the length to
the rationals in the open intervals between those points,
then recursing down. (I'll be more specific below.)

Some questions I have:
−− I can show that, if all these lengths have a sum, that
sum is epsilon; how would I show they have a sum? (Remember
that these intervals are not distinct.)
−− This scheme looks fractal to me (speaking informally).
Is it fractal (speaking more formally)? What would its
dimension be? Does this hypothetical dimension > 0 have
implications for measure theory?

What I'm talking about:
Let e be an arbitrarily small positive number.
Let k, m, n be positive integers and p, q be rationals.
Let d(p) be the length of the interval covering the rational
p in (0, 1).

Define
d(1/n) = e/2^n, for n = 2,3,4,...
d(1/(n+p)) = 1/2^(n+1)*d(p), for n = 1,2,3,... and 0 < p < 1

If SUM_{ 0 < p < 1}(d(p)) exists, then

SUM_{ 1/(n+1) < p < 1/n}(d(p)) =

SUM_{ 0 < q < 1}(d(1/(n+q))) =

1/2^(n+1)*SUM_{ 0 < q < 1 }(d(q))

SUM_{ 0 < p < 1}(d(p)) =

SUM_{ n = 2,3,4,...}(d(1/n)) +
SUM_{ n = 1,2,3,...}(
SUM_{ 1/(n+1) < p < 1/n }(d(p))) =

SUM_{ n = 2,3,4,...}(e/2^n) +
SUM_{ n = 1,2,3,...}(1/2^(n+1))*
SUM_{ 0 < q < 1}(d(q)) =

e/2 + 1/2*SUM_{ 0 < q < 1}(d(q))

SUM_{ 0 < p < 1}(d(p)) = e/2 + 1/2*SUM_{ 0 < q < 1}(d(q))

SUM_{ 0 < p < 1}(d(p)) = e

Then, if the sum over the rationals p in (0, 1)
of the lengths of intervals d(p) converges, that sum is e.

Jim Burns
.