Re: sin x / x tends to 1...

Source: http://sci.tech−archive.net/Archive/sci.math/2005−09/msg00630.html

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- *Date*: Sat, 03 Sep 2005 11:18:58 −0500

On Sat, 3 Sep 2005 13:22:46 +0000 (UTC), Darren J Wilkinson
<d.j.wilkinson@xxxxxxxxx> wrote:

>I've a question about the limit of sin x / x as x tends to zero. Of
>course, it's 1 (I think), but I've never seen a satisfactory proof. The
>proof I was given, and the proofs I can find in standard texts all rely
>on knowing the area of a circular sector. However, to know the area of a
>circular sector, one must know the area of a circle.

Not really. Why not? See a few paragraphs down, after some disclaimers:

First I should say that it seems to me that the simplest thing really is to define sin(x) by the power series. Then the proof that sin(x)/x −> 1 is very simple (it's not clear to me whether you said in a post below that this was not clear to you), and it's not too hard to show that this turns out to be the same as a geometric definition.

But if you say that's not elementary enough fine. Let's talk about sin defined "geometrically", and let's ignore the fact that just _defining_ what the area of a region _is_ takes some not−so−elementary work; let's assume, as in a typical elementary context, that we know what areas are.

(Um, when I say let's assume we know what area is I don't mean we're going to assume we know that the area of a circle of radius 1 is pi, I just mean that we're going to assume that we know what the _notion_ of "area" means.)

Ok. We do have to assume that the area of a circular sector is proportional to the angle. To prove that we'd first have to define "area", and that would take us out of absolutely elementary things. It's certainly intuitively plausible from our intuitive notion of "area".

We assume that the area of a circular sector is proportional to the angle. We define the sine of an _angle_ geometrically. We define a radian to be that angle such that a circular sector with radius 1 and opening one radian has area 1.
If we're justifying things then the existence of such an angle depends on continuity and the intermediate value theorem. But we're not worrying about that, lest things become non−elementary − the point is we define one radian in terms of a certain _area_, instead of in terms of a certain arc length. If you're worried about how to show that that's the same as a radian defined in terms of arc length don't worry about that − say it's "aradian" defined in terms of area and "radian" defined in terms of arc length − we won't need the fact that 1 aradian = 1 radian.

Now we define the sine of an _angle_ geometrically, and then if x is a _number_ we say that sin(x) is the sine of x radians. Now draw that picture you see in calculus books.

That picture contains a little triangle contained in a circular sector, which is contained in a larger triangle. If you consider the areas of those three sets you see

sin(x) cos(x) <= x <= tan(x)

(the sector has area x by our definition of "radian"!) Hence

cos(x) <= sin(x)/x <= 1/cos(x)

and we're done.

There's a lot there that was not "rigorous" − you said you wanted elementary. But the sort of, um, circularity you're concerned with here simply doesn't come up − we didn't need to know the area of a circle.

(We could show the area of a circle was pi r^2 by defining pi to be the number of radians in 180 degrees. Now to show that the circumference is what it is we finally need to show that "aradians" are the same as radians...)

>All the derivations

>I know for the area of a circle make use (either directly or indirectly)
>on the sin x / x limit, and there lies my disatisfaction. Of course it's
>easy to get the upper bound of one, and I'm happy to use the area
>argument to establish the existance of a limit. However, it seems to be
>surprisingly awkward to establish the obvious lower bounds (such as cos
>x) using elementary arguments. Does anyone know a nice proof?
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>Regards,

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David C. Ullrich
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    - *From:* Ignacio Larrosa Cañestro
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    - *From:* Lee Rudolph
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