Re: sin x / x tends to 1...

Source: http://sci.tech−archive.net/Archive/sci.math/2005−09/msg00822.html

---

- *From*: David C. Ullrich <ullrich@xxxxxxxxxxxxxxxx>
- *Date*: Sun, 04 Sep 2005 08:32:30 −0500

On Sat, 3 Sep 2005 21:17:29 +0000 (UTC), Darren J Wilkinson
<d.j.wilkinson@xxxxxxxxx> wrote:

>David C. Ullrich <ullrich@xxxxxxxxxxxxxxxx> wrote:
>> >Now, if I understood it correctly, you showed that sin x / x tended to
>> >one when x was measured in "aradians".
>>
>> You might note that there was a factor of 1/2 missing in my
>> definition of "aradian", which was cancelled by the missing
>> 1/2 in the formula I was using for the area of a triangle.
>> (Details... heh.)
>
>Yes, of course − easily fixed.
>
>> >Is there an easy way to show that
>> >aradians and radians are the same?
>>
>> In the sense in which I would use the word "show",
>> I doubt it − I don't think that we can even give an
>> easy definition of area, much less arc length.
>
>That's what I thought... As I said in my original post, I'm quite happy
>using the area argument to establish the existance of a limit on [0,1],
>but I'd like to know that it is 1... ;−)
>
>OK, so I'm starting to give up on a neat proof for a bright 15 year old.
>So, how about one for someone with a degree in mathematics and a PhD in
>theoretical statistics... ;−)

That's easy. The actual _definition_ of the length of a curve
is the supremum of the lengths of inscribed polygons, and using
that it's easy to show that the circumference of the unit circle
is twice the area.

>Assuming a power series definition of
>sin x, that we can call psin x, the limit is (of course) obvious. Is
>it very easy to show that psin x = sin x, where sin x is the usual
>geometric definition (with regular radians)?

That's also not too hard − it's not included in the place in Rudin where he works out the basic properties of trig functions starting from power series, but it's in one of the appendices to my complex notes, to be published some day when I get around to it:

Start with the definition in terms of power series.
It's easy to show that sin' = cos and cos' = −sin.
It follows from that that sin^2 + cos^2 is constant,
and then looking at x = 0 shows that sin^2 + cos^2 = 1.

Now it's not hard to show that cos(3) < 0 by evaluting the sum of the first three terms and noting that the sum of the rest of the terms is negative (group them in pairs and use some easy inequalities.) So cos has a smallest positive zero; define pi to be twice the smallest positive zero of cos.

Now cos > 0 on (0,pi/2), so sin is increasing there, hence sin(pi/2) > 0, so sin^2 + cos^2 = 1 shows that sin(pi/2) = 1. Now sin > 0 on (0, pi/2), so cos is decreasing on that interval.

Now define c(t) = (cos(t), sin(t)). Say I = [0,pi/2]. We've shown that c is a (continuous) mapping of I into the first quadrant, and that c is 1−1 on I.

Oops, back up. Define exp(z) (for complex z) by the power series. Show exp' = exp. It follows that exp(z) exp(−z) is constant, and z = 0 shows that the constant is 1. So exp has no zero.
Fix w and define f(z) = exp(z+w)/exp(z). It follows that f is constant, and then z = 0 shows that exp(z+w) = exp(z) exp(w). This formula implies the addition formulas for sin and cos.

Back to our story. We've shown that c is a ontinuous mapping from I to the first quadrant of the unit circle. Since c(0) = (1,0),
c(pi/2) = (0,1) and c(I) is connected, it follows that c maps I 1−1 and onto the first quadrant of the unit circle.

Now the addition formulas for sin and cos show that c maps the interval [pi/2, pi]
1−1 onto the second quadrant of the unit circle. Etc − now we know that c maps
[0,2pi] onto the unit circle, and that
c is 1−1 on [0,2pi] except for the fact that c(2pi) = c(0).

Now, c(t) = (cos(t), sin(t)). So to show
that cos and sin are the same as the
geometric definition we just need to show
that (for 0 < x < 2pi, say) the length
of the arc of the unit circle from (1,0)
to c(x) is exactly x. But this is immediate
from the calculus formula for the length
of a (paramtrized) curve: ||c'(t)|| = 1,
so the length of this arc is the integral
of ||c'(t)|| from 0 to x, which is is
exactly x. qed.

****

David C. Ullrich
.

- ***Follow−Ups***:
    - ***Re: sin x / x tends to 1...***
        - ◊ *From:* Darren J Wilkinson

- ***References***:
    - ***sin x / x tends to 1...***
        - ◊ *From:* Darren J Wilkinson
    - ***Re: sin x / x tends to 1...***
        - ◊ *From:* David C . Ullrich
    - ***Re: sin x / x tends to 1...***
        - ◊ *From:* Darren J Wilkinson
    - ***Re: sin x / x tends to 1...***
        - ◊ *From:* David C . Ullrich
    - ***Re: sin x / x tends to 1...***
        - ◊ *From:* Darren J Wilkinson

- Prev by Date: ***Re: Tensor products of some Banach spaces***
- Next by Date: ***Re: sin x / x tends to 1...***
- Previous by thread: ***Re: sin x / x tends to 1...***
- Next by thread: ***Re: sin x / x tends to 1...***
- Index(es):
    - ***Date***
    - ***Thread***