Locus of z(z+c) for |z|=1 (z complex) and an optical phenomenon

Source: http://sci.tech−archive.net/Archive/sci.math/2005−10/msg01745.html

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- *Date*: Mon, 17 Oct 2005 00:52:20 GMT

I was motivated to consider the following subject by the analysis
of an optical phenomenon I observed by chance and then found
its cause ... see below

What can be said about the image of the unit circle |z|=1 (z complex
number) by the map f: z |−> z * (z+c), c a complex constant ? In fact,
we don't loose soemthing essential by taking c real and >= 0
(indeed, let t be any complex nb. of modulus 1; replacing z by t*z
which is just a reparametrization of the unit circle and then
multiplying f(z) by t^(−2) which is a rotation about the center 0 will
do the same as replacing c by c*t^(−1), and t may be chosen such
that this is real and >= 0)

Considering f(z)+1 = z^2+c*z+1 instead of f(z) actually simplifies
things, this is just a change of origin in the plane. For c=0 the
the said image − I'll just call it 'my curve' − can be considered as
the unit circle "taken twice". For c=2 (f(z)+1 = (z+1)^2) we get
the curve called a cardioid (heart curve) − this I recognized
from its shape and then found in an old book where it is described:
that the general case c>0 gives in fact a type of curve (which
I had forgotten) known as a "limaçon", more precisely called
limaçon of Pascal as I discovered looking for it in Wikipedia (fr).
Taking x,y as the real and imag. parts of w=f(z)+1, its equation is
[E] (x^2+y^2−2*x) ^ 2 = c^2 * (x^2+y^2)
or in polar coordinates r = c+2*cos(phi) where w=r*exp(i*phi), i^2=−1.

The definition I found in the old book is quite different from
mine, but one can easily verify that all definitions / formulas
encountered are "essentially" equivalent. A external link in Wikipedia
led me to the following (French) site explaining the curve in detail:
http://www.mathcurve.com/courbes2d/limacon/limacon.shtml
there the different possible shapes are shown (with pictures) and
there is a good discussion of its properties etc. there, so that
I drop many details which I intended to include in this post ...

In fact there is a little problem with [E] when c>2: this equation
is always satisfied by the origin x=y=0, but for c>2 this is an
isolated point of the (real) algebraic 'curve' defined by [E] which

the polar coord. version and my definition above don't reach.
This is why I say that the def. / formulas are (only) "essentially" equivalent ...
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Now the optical phenomenon behind this all. Consider the situation: there is a small short cylinder (a normal one with circular sections in planes perpendicular to the generatrices, i.e. obtainable by the rotation of a straight line around an axis parallel to it), where by 'short' is meant that it is limited by two near circular sections. Length and diameter are both approx. 3 cm. (or 1 inch) , the axis is horizontal. The room where it is has (among others) a wall perpendicular to the cylinder axis, the wall being at about 85 cm. (or a yard) of distance from the cylinder (which is at a hight of about 1m / 1 yard above the ground). A strong light source at 'infinity' (the sun in a blue sky) shines through a window into the room, obliquely (angles with the cylinder axis measured in horizontal as well as vertical planes >0 but <30 deg. − estimated).

The cylinder is well shaped, metallic and smooth, so that it reflects light after the classical law that a ray is reflected in a ray that is symmetric to it with respect to the normal line to the reflecting surface at the point where the ray meets the surface. This produces on the wall a nearly complete annulus (= the area between two concentric circles), rather narrow − so that it was just a circle for me first − much brighter than the surrounding wall surface and of large dimension (may−be with a diameter of more than 1m / yard).

Is the 'projection' of the cylinder to the wall an exact circle (when the cylinder is very short, so that we may neglect its length) ?
It probably cannot be a complete circle because the external light will not reach the whole cylinders and some reflected rays might not be sent to the wall (I'm not sure about this 2d fact) − but my analysis showed that already half of the cylinder surface might give a complete circle / annulus. By my analysis of this thing by partly geometrical, partly algebraic means, I found this: yes, if we can neglect its diameter (and its center will even be at the intersection of the axis and the wall). But if we don't want to do this, then − after some rotation and resizing − the projection is "my" curve. (From the result with a 'neglected diameter' it was easy to go to the result without this neglecting. In this step I found it helpful to use complex numbers to treat the plane geometry involved.)

BTW the precise result I got for the part where I neglect somehow the diameter suggests that it might be possible to obtain it in a more purely 'geometric' way. I wonder if someone can produce such an argument: my result says that a 'naive simplification' that 'can only be wrong' a priori produces a correct result ... a vague hint to what I mean here: although the plane spanned by a ray from the source and the normal to the cylinder − containing also the reflected ray − is in 'general position', what happens on the wall results from the orthogonal projection to the wall of all 3 directions

as if the said plane was the wall ... up to a constant factor (for
the vectors involved) not depending on the point where reflection
takes place)

Another remark: the website mentioned above and others connected
with it also contain very similar considerations about reflecting
(surfaces or) curves, but I nevertheless find it difficult to see
a direct connection with the phenomenon I describe − which is
"irreducibly" 3D whereas those discuss practically only 2D things.
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