

Transcendence degree of Coeff f

Source: <http://sci.tech-archive.net/Archive/sci.math/2006-01/msg02416.html>

- *From:* viz <viz@xxxxxxxxxxxxxxxxxx>
 - *Date:* Wed, 18 Jan 2006 10:13:45 +0100
-

Let $X=\{X_1, \dots, X_n\}$, $Y=\{Y_1, \dots, Y_n\}$ and $f, g \in k[X, Y]$ polynomials (k is a field of characteristic 0). Let $\text{Coeff}_Y f$ be the field generated over k by the coefficients of f seen as function of the Y 's.

Prove or disprove:

$$\text{tr deg}_k \text{Coeff}_Y f.g \geq \text{tr deg}_k \text{Coeff}_Y f - 1$$

where tr deg denotes the transcendence degree of the field extension $k/\text{Coeff}_Y f$.

Note that the -1 is needed, as this example shows:

$$f = X_1.Y_1 + X_1^2.X_2.Y_2, g = X_2$$

If g is only a polynomial of the X 's, the claim is true. Proof: use the fact that the transcendence degree of $k/k(f_1, \dots, f_n)$ (f_1, \dots, f_n being the coefficients of f) is equal the rank of the matrix, whose columns are given by the gradients of the f 's. Then simply apply the product rule of derivation and the property $\text{rank}(A + B) \geq \text{rank}(A) - \text{rank}(B)$.

I have somehow the feeling, that only polynomials $g(X, Y)$ of the form $g(X, Y) = g_1(X).g_2(Y)$ can indeed decrease the transcendence degree and that for every other polynomial we really have:

$$\text{tr deg}_k \text{Coeff}_Y f.g \geq \text{tr deg}_k \text{Coeff}_Y f$$

Any ideas about a proof (or, in the worst case, a counterexample)?

Transcendence degree of Coeff f

Thanks a lot.