Every solution of diff.eq. y'=((y^2+1)/(x^4+1))^(1/3) has two hor. asymptotes? How to prove that?

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How to prove that every solution of differential equation

(*) y'=((y^2+1)/(x^4+1))^(1/3)

has two horizontal asymptotes?

I tried to apply the following:

LEMMA: Let f1(x,y) and f2(x,y) be continuous and Liphsitz for all (x,y)
in G, where
G = {(x,y): x in (a, b), y in (c, d)}.
Suppose that f1(x,y) < f2(x,y) in G.
Let y1(x), y2(x) for x in (a1, b1) (a1>=a, b1<=b) are solution
of differential equations:
y1' = f1(x,y1) and y2' = f2(x,y2) respectively such that
y1(x0)=y2(x0)=y0
(of course (x0,y0) in G).
Then y1(x) <= y2(x) for all x in (x0,b1) (x0 in (a1, b1)).

P r o o f. y1(x) and y2(x) are unique and continuously differentiable
on (a1, b1)
so the function h(x) = y2'(x) − y1'(x) is continuous on (a1, b1) and
since
y1(x0) = y2(x0) it follows that h(x0) = y2'(x0) − y1'(x0) = f2(x0,y0) −
f1(x0, y0) >0.
By continuity of h we have that h(x) > 0 in some interval (x0−r, x0+r),
where
r>0, a1<=x0−r, x0+r<=b1. Hence y2(x)−y1(x) = int(h(t), t=x0..x) > 0
for x in (x0, x0+r). (int(h,x=x1..x2) means definite integral of
function h...).

Now suppose that there exists p in (x0, b1) such that y1(p)>y2(p).
Let p0 = inf{p in (x0, b1): y1(p)>y2(p)}.
We have p0>=x0+r and also y1(x)<y2(x) for x in (x0, p0) so
we must have y1(p0) = y2(p0)=yp0 since y1 and y2 are continuous.

It follows that h(p0) = y2'(p0) − y1'(p0) = f2(p0,yp0) − f1(p0, yp0)

0.

By continuity of h we have that h(x) > 0 in some interval (p0−r1,
p0+r1), where
r1>0, a1<=p0−r1, p0+r1<=b1. Thus we have that h(x)>0 for all x in (p0,
p0+r1).
Hence y2(x)−y1(x) = int(h(t), t=p0..x) > 0 for x in (p0, p0+r1) and
this contradicts the definition of p0.
E n d o f t h e p r o o f.

Because f1(x, Y) = ((y^2+1)/(x^4+1))^(1/3) > 0 for all (x,y) in R^2
so every solution of the differential equation (*) is increasing.
Hence it is enough to show that the solution is bounded above.

Suppose that y1(x) is a maximal solution of (*) defined for all x in
(−oo, oo)
(I shall show this later).

If the solution y1(x) <=1 for al x we are done.
Suppose y1(x) > 1 for x in some interval (a, infinity), a >0.
Now let f1(x, Y) = ((y^2+1)/(x^4+1))^(1/3) and
f2(x, Y) = (2y^2/(x^4)^(1/3). Let y2(x) = (− (2^(1/3))*x^(−1/3)+C
)^3 where
C is constant such that y1(x0)=y2(x0)=y0.
Consider x0>x1>0.
We have f1(x,y) < f2(x,y) for all x>x1>0 and y>1. By LEMMA
we have that y1(x) <= y2(x) for x >= x0. Since y2(x) is bounded above
for x>=x1
so is y1(x).

Similarly y1(x) <− 1 for x in some interval (−infinity, a), a <0.

Now why every maximal solution y1(x) of (*) is defined for all x in
(−oo, oo)?
Because y1(x) is bounded and the curve (x, y1(x)) must reach the
boundary
of the set R^2 so x must reach +oo and −oo.

Does anybody have some hint, some different (especially simpler)
solution
of the problem?

TIA

Jan

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