Fourier Transform, Smooth Functions

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I have two questions about the relationship between the smoothness of a function and its Fourier transform. I know that if a function is very smooth, then its Fourier transform can decay faster than 1/s^p for any p>0. But, can its Fourier transform decay faster than that if the function is zero outside of [−1,1]? More precisely,

1) Does there exist a function f(x) such that f is continuous, f(x) is zero when abs(x) > 1, and its Fourier transform, g(s), is order exp(− abs(s)^p) where p > 1/2?

2) What is the largest value of p such that there exists a function f such that f is continuous, f(x) is zero when abs(x) > 1, and its Fourier transform, g(s), is order exp(− abs(s)^p)?

If you understand the questions, then you can stop reading here and reply with your thoughts. For everyone else, I will try to define the terms in those questions.

Given a function f(x) that maps reals to reals. Assume that f(x) has the following properties:

1) f(x) is continuous,
2) f(x) is 0 when x<−1 or x>1, and
3) 0<= f(x) <= 1 for all x.

Define the Fourier transform of f, to be the function g that maps reals into complex numbers with the formula

g(s) = Integral[f(x)* exp(− 2 * pi * i * x * s), {x, −Infinity, Infinity}].

Define the "set of functions of order h(s)", denoted O(h(s)), to be the set of all complex valued functions w(s) with one real argument such that there exists a real constant c obeying

abs(w(s)) < c h(s) for all s.

How quickly can g(s) decay?

Or more precisely, does there exist a function f(x) obeying the assumptions 1 − 3, such that its Fourier transform g(s) is contained in the set O(exp(− abs(s)^p)) for p = 1/4? How about for p=1/2, 1, or 2?

I don't know the answer, but I suspect that the answer is yes for p<0.5 and no for p>1.

Cheers,
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