

Re: Fourier Transform, Smooth Functions

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On 25 Feb 2006 08:22:58 -0800, "irchans" <infinitemath@xxxxxxxx> wrote:

I have two questions about the relationship between the smoothness of a function and its Fourier transform. I know that if a function is very smooth, then its Fourier transform can decay faster than $1/s^p$ for any $p > 0$. But, can its Fourier transform decay faster than that if the function is zero outside of $[-1, 1]$? More precisely,

- 1) Does there exist a function $f(x)$ such that f is continuous, $f(x)$ is zero when $\text{abs}(x) > 1$, and its Fourier transform, $g(s)$, is order $\exp(-\text{abs}(s)^p)$ where $p > 1/2$?
- 2) What is the largest value of p such that there exists a function f such that f is continuous, $f(x)$ is zero when $\text{abs}(x) > 1$, and its Fourier transform, $g(s)$, is order $\exp(-\text{abs}(s)^p)$?

I'm not sure. My first guess was that $p = 0$ was the largest possible, but a proof of that that I expected should work does not work.

It's easy to see that $p = 1$ is not possible: If $p = 1$ then it's easy to see that f must extend to a function holomorphic in a horizontal strip containing the real axis, and hence f (restricted to the line) cannot have compact support, unless of course $f = 0$.

That's so easy that it seems like a stronger result must be available with more work. I had a thought – perhaps the Denjoy–Carleman theorem (see for example "quasi-analytic classes" in Rudin "Real and Complex Analysis") would rule out the case $0 < p < 1$. But it doesn't, if I did the calculations correctly: If you assume that g satisfies the given bound that gives a bound on the size of the n -th derivative of f . A change of variables shows that $D^n f$ is bounded by something like $\Gamma(n/p)$. But if $p < 1$ then Stirling's formula shows that $\sum (1/\Gamma(n/p))^{1/n}$ is finite, so the class of f satisfying those bounds on the derivatives is not quasi-analytic.

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So I'm really not sure. I should know this... My guess is that the best possible p is either 0 (that still seems right to me, although I can't say why) or $1 - \epsilon$, as the D-C theorem would tend to be indicating.

(You should check my work – if I made an error in the calculations then the D-C theorem does show that $p = 0$ is the largest possible. On the other hand it seems possible that I did the calculations correctly, because what I did with the D-C theorem does give a harder proof that the case $p = 1$ is impossible...)

If you understand the questions, then you can stop reading here and reply with your thoughts. For everyone else, I will try to define the terms in those questions.

Given a function $f(x)$ that maps reals to reals. Assume that $f(x)$ has the following properties:

- 1) $f(x)$ is continuous,
- 2) $f(x)$ is 0 when $x < -1$ or $x > 1$, and
- 3) $0 \leq f(x) \leq 1$ for all x .

Define the Fourier transform of f , to be the function g that maps reals into complex numbers with the formula

$$g(s) = \text{Integral}[f(x) * \exp(-2 * \pi * i * x * s), \{x, -\text{Infinity}, \text{Infinity}\}].$$

Define the "set of functions of order $h(s)$ ", denoted $O(h(s))$, to be the set of all complex valued functions $w(s)$ with one real argument such that there exists a real constant c obeying

$$\text{abs}(w(s)) < c h(s) \text{ for all } s.$$

How quickly can $g(s)$ decay?

Or more precisely, does there exist a function $f(x)$ obeying the assumptions 1 – 3, such that its Fourier transform $g(s)$ is contained in the set $O(\exp(-\text{abs}(s)^p))$ for $p = 1/4$? How about for $p = 1/2$, 1, or 2?

I don't know the answer, but I suspect that the answer is yes for $p < 0.5$ and no for $p > 1$.

Cheers,
Irchans

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