Re: Fourier Transform, Smooth Functions

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irchans wrote:

> I have two questions about the relationship between the smoothness of a function and its Fourier transform. I know that if a function is very smooth, then its Fourier transform can decay faster than 1/s^p for any p>0. But, can its Fourier transform decay faster than that if the function is zero outside of [−1,1]? [... snip ...]

I can only guess what you really want, but here is my two cents worth.

A Fourier transform can even decay infinitely fast. For example, if the original function is a simple constant, then the Fourier transform is a delta function. You can see this if you determine the Fourier transform of a rectangle: f(x) = 0 for x < −a/2, f(x) = 1 for −a/2 < x < +a/2 and f(x) = 0 for x > +a/2. Then F(s) = a/pi.sinc(a/2.s) [sinc(x) = sin(x)/x for x <> 0 and sinc(0) = 1] The integral F(s).ds from −oo to +oo equals

= 1 independent of a . Consider the case where a becomes large −> oo. Then the rectangle f(x) becomes a constant 1 and the Fourier transform F(s) becomes the delta function (iff I made no mistakes). The reverse is also true. Therefore a "wide" function in the original domain causes a "narrow" function in the Fourier domein and vice versa.

Restricting your original f(x) to an interval [−1,+1] makes its Fourier transform F(s) less decaying. There is a general theorem in the theory which is very much alike (in fact it _is_) the Heisenberg uncertainity principle. It sounds like the following. Form the variance dx^2 of the original function as the integral (−oo,+oo) x^2 f(x) dx , the variance ds^2 of the Fourier transform as integral (−oo,+oo) s^2 F(s) ds / (2Pi)

apart from norming constants. Then there is the theorem: dt.ds > 1/2 , Which is equivalent to Heisenberg's uncertainity principle in quantum mechanics (!) Here the limiting value dt.ds = 1/2 is reached if we take

f(x) = exp(−(x/sigma)^2/2)/(sigma.sqrt(2.pi)), F(s) = exp(−(s.sigma)^2)
a combination which is "optimal" in the sense that dx.ds is minimal.

Hope this helps.

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