Re: Fourier Transform, Smooth Functions

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On 26 Feb 2006 17:13:31 −0800, "david petry"
<david_lawrence_petry@xxxxxxxxx> wrote:

> irchans wrote:
>
>> I have two questions about the relationship between the smoothness of a function and its Fourier transform. I know that if a function is very smooth, then its Fourier transform can decay faster than 1/s^p for any p>0. But, can its Fourier transform decay faster than that if the function is zero outside of [−1,1]? More precisely,
>>
>> 1) Does there exist a function f(x) such that f is continuous, f(x) is zero when abs(x) > 1, and its Fourier transform, g(s), is order exp(− abs(s)^p) where p > 1/2?
>>
>> 2) What is the largest value of p such that there exists a function f such that f is continuous, f(x) is zero when abs(x) > 1, and its Fourier transform, g(s), is order exp(− abs(s)^p)?
>
> Here's a way to produce functions f(x) such that g(s) is especially rapidly decreasing.
>
> Let {a_k} be an infinite sequence of positive reals such that sum(a_k) < oo, and let it be such that the sum converges very very slowly. Then look at the function g(s) = product sin(a_k s)/(a_k s). Let f(x) be the Fourier transform of g(s).
>
> Then f(x) will have compact support,

And f will be infinitely differentiable.

> and g(s) will decay faster than
> 1/s^p for all p. Now you want to find something that decays even faster. I contend, without giving a proof, that if the series {a_k} is

> well chosen and the sum converges slowly enough, then we can ensure that g(s) will decay more rapidly than exp(−s^m) for any particular m < 1.
>
> Unfortunately, I'm not absolutely certain I'm right. I remember looking at this problem years ago, and the above is how I remember solving it.

I believe this may very well be right. Here's a sloppy version of a proof (various statements of the form a ~ b would need to be converted to explicit inequalities to make an actual proof):

First note that there exists c > 0 such that if |t| < 1 then

(i) sin(t)/t <= exp(−c t^2).

(Note that "c" below will denote various constants, differing from line to line.)

Now suppose that m < 1. Choose eps > 0 so that

(ii) 2 − m > (1 + 2 eps)/(1 + eps)

(such an eps > 0 exists since 2 − m > 1).

Let a_k = 1/k^(1+eps).

(Note that the argument must use the fact that m < 1, since m = 1 is impossible, as I showed in another post. Here we see where m < 1 is used; we need sum a_k < infinity to give f compact support, and that requires eps > 0, hence requires m < 1. The fact that we can see where m < 1 is used seems to me to lend some credibility...)

Suppose that s is large. Choose N so that s ~ 1/a_N. Now (i) shows that

(iii) g(s) <= exp(−c sum_{k=N}^infinity a_k^2 s^2).

Our choice of a_k shows that

sum_{k=N}^infinity a_k^2 ~ 1/N^(1 + 2eps)

= a_N ^ ((1 + 2 eps)/(1 + eps))

So (ii) gives

sum_{k=N}^infinity a_k^2 > c a_N ^ (2 − m) ~ s^(m−2),

so

sum_{k=N}^infinity a_k^2 s^2 > c s^m,

hence (iii) shows

g(s) <= exp(−c s^m).

Keen.

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David C. Ullrich
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